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TECHNICAL APPENDIX TO "WHY DOES INVENTORY INVESTMENT FLUCTUATE SO MUCH?"

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1. Solving the LQ Approximate Problem Under Complete State Information

This Appendix describes the algorithm used to find decision rules for i^*_t, k^*_t, and h^*_t that solve the linear quadratic approximation to the model in the paper. For most of the discussion, I assume there is full state information, in the sense that $v_t \equiv 0$. At the end I show how to modify the complete state information solution to apply in the incomplete state case. The solution strategy is to first transform the problem into the form of the linear regulator problem in the engineering literature. (See, e.g., Kwakernaak and Sivan [1972].) Define

$$s_t = \begin{pmatrix} k^*_{t-1} \\ i^*_{t-1} \\ w_t \\ w_{t-1} \end{pmatrix}, \quad d_t = \begin{pmatrix} i^*_t \\ k^*_t \\ h^*_t \end{pmatrix}, \quad \phi_1 = \begin{bmatrix} \underset{2\tilde{\times}2}{0} & \underset{2\tilde{\times}2}{0} & \underset{2\tilde{\times}2}{0} \\ \underset{2\tilde{\times}2}{0} & \underset{2\tilde{\times}2}{A} & \underset{2\tilde{\times}2}{0} \\ \underset{2\tilde{\times}2}{0} & \underset{2\tilde{\times}2}{I} & \underset{2\tilde{\times}2}{0} \end{bmatrix}, \quad \phi_0 = \begin{pmatrix} \underset{2\tilde{\times}2}{0} \\ \underset{2\times 1}{a} \\ \underset{2\times 1}{0} \end{pmatrix}$$

$$(1.1) \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \underset{4\tilde{\times}1}{0} & \underset{4\tilde{\times}1}{0} & \underset{4\tilde{\times}1}{0} \end{bmatrix}, \quad e_t = \begin{pmatrix} \underset{2\tilde{\times}1}{0} \\ \underset{2\times 1}{\varepsilon_t} \\ \underset{2\tilde{\times}1}{0} \end{pmatrix}, \quad Ee_te_t^T = W = \begin{bmatrix} \underset{2\tilde{\times}2}{0} & \underset{2\tilde{\times}2}{0} & \underset{2\tilde{\times}2}{0} \\ \underset{2\tilde{\times}2}{0} & V & \underset{2\tilde{\times}2}{0} \\ \underset{2\tilde{\times}2}{0} & \underset{2\tilde{\times}2}{0} & \underset{2\tilde{\times}2}{0} \end{bmatrix} .$$

The return function may be written

$$(1.2) \qquad R(s_t, d_t) = c + c_1^T d_t + c_2^T s_t + s_t^T R s_t + d_t^T Q d_t + 2 s_t^T F d_t .$$

Here, c is a scalar, $c_1 \sim 3 \times 1$, $c_2 \sim 6 \times 1$, $R \sim 6 \times 6$, $Q \sim 3 \times 3$, $F \sim 6 \times 3$. Also,

$$(1.3) \qquad R = \frac{1}{2} \begin{bmatrix} r_{11} & r_{13} & r_{17} & r_{16} \\ r_{31} & r_{33} & r_{37} & r_{36} \\ r_{71} & r_{73} & r_{77} & r_{76} \\ r_{61} & r_{63} & r_{77} & r_{66} \end{bmatrix}, \quad Q = \frac{1}{2} \begin{bmatrix} r_{44} & r_{42} & r_{45} \\ r_{44} & r_{22} & r_{25} \\ r_{54} & r_{52} & r_{55} \end{bmatrix}$$

$$F = \frac{1}{2}\begin{bmatrix} r_{14} & r_{12} & r_{15} \\ r_{34} & r_{32} & r_{35} \\ r_{74} & r_{72} & r_{75} \\ r_{64} & r_{62} & r_{65} \end{bmatrix}.$$

Here, r_{ij} denotes the cross derivative of r in the paper's equation (3.5) with respect to its ith and jth arguments, evaluated at $k^*_{t-1} = k^*_t = k^*_s$, $i^*_{t-1} = i^*_t = i^*_s$, $h^*_t = h^*_s$, $w_t = w_s$. Note that r_{ij} are scalars for all i, j = 1, ..., 5. However, r_{kj} and r_{ik} are 2 × 1 and 1 × 2, respectively, k = 6, 7, and r_{kk} is 2 × 2, k = 6, 7.

Finally, consider the constant terms, c_1 and c_2. Let $z_1 = (r_4,r_2,r_5)^T$, $z_2 = (r_1,r_3,r_7^T,r_6^T)^T$. Here, r_j denotes the derivative of r with respect to its jth argument, evaluated at $k^*_{t-1} = k^*_t = k^*_s$, $h^*_t = h^*_s$, $w_{t-1} = w_t = w$. Note that r_7 and r_6 are 2 x 1 vectors, while r_j, j = 1, ..., 5 are scalars. Then,

$$c_1 = z_1 - 2F^Ts - 2Qd$$

$$c_2 = z_2 - 2Fd - 2Rs,$$

where s and d are the steady state values of s_t and d_t, respectively. In particular,

$$s = \begin{pmatrix} k^*_s \\ i^*_s \\ w \\ w \end{pmatrix}, \quad d = \begin{pmatrix} i^*_s \\ k^*_s \\ h^*_s \end{pmatrix}.$$

Analytic formulas for the r_j's and r_{ij}'s are provided in section 2.

In these terms, the LQ approximate problem is to choose a contingency plan for d_t to maximize

$$(1.4)\qquad E_0 \sum_{t=0}^{\infty} \beta^t\{c+c_1^T d_t+c_2^T s_t+s_t^T R s_t+d_t^T Q d_t+2s_t^T F d_t\}$$

subject to

$$(1.5)\qquad s_{t+1} = \phi_0 + \phi_1 s_t + B d_t + e_{t+1}.$$

The solution to this problem is obtained by iterating on the following functional equation in v:

$$(1.6)\qquad v'(s_t) = \max_{d_t}\{R(s_t,d_t)+\beta E_t v(s_{t+1})\}$$

subject to (1.5) and s_t given and observable. Here,

$$(1.7)\qquad v(s_t) = v_c + v_s^T s_t + s_t^T v_Q s_t.$$

I now describe one step in this iteration. Substituting (1.7) into (1.6), get

$$(1.8)\qquad v'(s_t) = \max_{d_t}\{\tilde{c}+\tilde{c}_1^T d_t+\tilde{c}_2^T s_t+s_t^T\tilde{R}s_t+d_t^T\tilde{Q}d_t+2s_t^T\tilde{F}d_t\},$$

where

$$(1.9a)\qquad \tilde{c} = c + \beta v_c + \beta v_s^T \phi_0 + \beta\phi_0^T v_Q \phi_0 + \beta \operatorname{tr}(v_Q W)$$

$$(1.9b)\qquad \tilde{c}_1^T = c_1^T + \beta v_s^T B + 2\beta\phi_0^T v_Q B$$

$$(1.9c)\qquad \tilde{c}_2^T = c_2^T + \beta v_s^T \phi_1 + 2\beta\phi_0^T v_Q \phi_1$$

$$(1.9d)\qquad \tilde{R} = R + \beta\phi_1^T v_Q \phi_1$$

$$(1.9e)\qquad \tilde{Q} = Q + \beta B^T v_Q B$$

$$(1.9f)\qquad \tilde{F} = F + \beta\phi_1^T v_Q B.$$

The solution to the maximization in (1.8) is

$$(1.10)\qquad d_t = K_0 + K_1 s_t,$$

where

(1.11a) $\quad K_0 = -\frac{1}{2}\tilde{Q}^{-1}\tilde{c}_1$

(1.11b) $\quad K_1 = -\tilde{Q}^{-1}\tilde{F}^T.$

Substituting (1.10) into (1.8) get

$$v'(s_t) = v_c' + (v'_s)^T s_t + s_t^T v_Q' s_t,$$

where

(1.12) $\quad v_c' = \tilde{c} + \tilde{c}_1^T K_0 + K_0^T \tilde{Q} K_0$

(1.13) $\quad v_s' = [\tilde{c}_1^T K_1 + \tilde{c}_2^T + 2K_0^T \tilde{Q} K_1 + 2K_0^T \tilde{F}^T]^T = K_1^T \tilde{c}_1 + \tilde{c}_2$

(1.14) $\quad v_Q' = \tilde{R} + K_1^T \tilde{Q} K_1 + 2\tilde{F}K_1 = \tilde{R} - \tilde{F}K_1 + 2\tilde{F}K_1 = \tilde{R} + \tilde{F}K_1.$

The solution to (1.4)-(1.5) is obtained by iterating on (1.6) to convergence.

The calculations just described can be simplified further by first iterating to convergence on v_Q and K_1 using (1.9d)-(1.9f), (1.11b), (1.14). Equations (1.9b), (1.9c), (1.11a), (1.13) can then be solved for v_s and K_0. The vector v_s is obtained by setting $v_s' = v_s$ in (1.13) and solving for v_s. This yields

$$v_s = [I-\beta(K_1^T B^T + \phi_1^T)]^{-1} \{K_1^T[c_1 + 2\beta B^T v_Q \phi_0] + c_2 + 2\beta\phi_1^T v_Q \phi_0\}.$$

The constant terms, $\tilde{c}$ and v_c, are not needed and so can be ignored. The solution to (1.6) when $v' = v$ solves (1.4)-(1.5).

An interesting feature of this problem is that the matrix Q is of rank 2. The results in section 3 show that

$$-Q = \ell\ell^T + pp^T,$$

where

$$\ell = [\tfrac{1}{2}\exp(-u)]^{1/2}\begin{pmatrix} r_4 \\ r_2 \\ \gamma \end{pmatrix}, \ p = \begin{pmatrix} 0 \\ 0 \\ [\tfrac{1}{2}\,\theta(1-\theta)\dfrac{\exp(u)}{c^*h^2}\,y^*]^{1/2} \end{pmatrix} .$$

A practical consequence of this is that the iterations on v_Q cannot be started at $v_Q = 0$, since in this case (1.11a) has no solution. Instead, I started v_Q at the product of the identity matrix and a small number.

The feedback rule in (1.10) expresses the decision variable as a function of the current state. It is convenient to express the rule for i_t^* as follows:

$$(1.14a) \qquad i_t^* = i(\tilde{d}_t, s_t),$$

where $\tilde{d}_t \equiv (k_t^*, h_t)^T$. The second two decision rules in (1.10) are written

$$(1.14b) \qquad h_t = h(s_t)$$

$$(1.14c) \qquad k_t^* = k(s_t).$$

To get (1.14a), carry out the maximization in (1.8) with respect to i_t^*, taking $\tilde{d}_t$ and s_t as given. Doing so, I get

$$(1.15) \qquad i_t^* = \alpha_0 + \alpha_1^T\tilde{d}_t + \alpha_2^T s_t \equiv i(\tilde{d}_t, s_t),$$

where

$$(1.16) \qquad \alpha_0 = -\tilde{c}_{11}/(2q_{11}), \ \alpha_1^T = -(q_{12}/q_{11}), \ \alpha_2^T = -\tilde{F}_1^T/q_{11},$$

$$\tilde{F} = [\underset{(6\times1)}{\tilde{F}_1} \,\vdots\, \underset{(6\times2)}{\tilde{F}_2}], \ \tilde{c}_1 = \begin{pmatrix} \underset{(1\times1)}{\tilde{c}_{11}} \\ \underset{(2\times1)}{\tilde{c}_{21}} \end{pmatrix}, \ \tilde{Q} = \begin{bmatrix} \underset{(1\times1)}{q_{11}} & \underset{(1\times2)}{q_{12}} \\ \underset{(2\times1)}{q_{12}^T} & \underset{(2\times2)}{q_{22}} \end{bmatrix} .$$

Thus, the solution to (1.4)-(1.5) under complete state information is given by (1.14)-(1.15).

It is convenient to express i_t^*, and $\tilde{d}_t$ in (1.14) in terms of ε_t, k_{t-1}^*, i_{t-1}^*. Doing so, we get

$$(1.17) \qquad i_t^* = i_0^* + i_1^* \begin{pmatrix} k_{t-1}^* \\ i_{t-1}^* \end{pmatrix} + i_w^* w_{t-1} + i_\varepsilon^* \varepsilon_t + i_d \tilde{d}_t,$$

where

$$i_0^* = (\alpha_0 + \alpha_{22}^T a),\ i_1^* = \alpha_{21}^T,\ i_w^* = (\alpha_{23}^T + \alpha_{22}^T A)$$

$$i_\varepsilon^* = \alpha_{22}^T,\ i_d^* = \alpha_1^T.$$

Here, α_2^T has been partitioned as follows:

$$\underset{1\times 6}{\alpha_2^T} \equiv (\underset{1\times 2}{\alpha_{21}^T} \ \vdots\ \underset{1\times 2}{\alpha_{22}^T} \ \vdots\ \underset{1\times 2}{\alpha_{23}^T}).$$

Also,

$$(1.18) \qquad \tilde{d}_t = d_0 + d_1 \begin{pmatrix} k_{t-1}^* \\ i_{t-1}^* \end{pmatrix} + d_w w_{t-1} + d_\varepsilon \varepsilon_t.$$

To describe the construction of d_0, d_1, d_w, d_ε, I first need some notation. Let $\bar{K}_0$ denote the vector formed by deleting the first element of K_0 and let $\bar{K}_1$ denote K_1 minus its first row. Then $\tilde{d}_t = \bar{K}_0 + \bar{K}_1 s_t$. Partition $\bar{K}_1$ as follows:

$$\underset{2x6}{\bar{K}_1} = \{\underset{2x2}{\bar{K}_1^{(1)}} \ \vdots\ \underset{2x2}{\bar{K}_1^{(2)}} \ \vdots\ \underset{2x2}{\bar{K}_1^{(3)}}\}$$

Then,

$$d_0 = \bar{K}_0 + \bar{K}_1^{(2)} a,\ d_1 = \bar{K}_1^{(1)},\ d_w = (\bar{K}_1^{(3)} + \bar{K}_1^{(2)} A)$$

$$d_\varepsilon = \bar{K}_1^{(2)}.$$

The solution to the incomplete information problem is obtained by replacing ε_t in (1.18) by $D(\varepsilon_t+\nu_t)$, when D is defined after (2.9) in the paper.

2. Analytic Formulas For Derivatives

The derivatives in (1.3) and those implied by c_1 and c_2 can be computed numerically and analytically. For checking purposes, it is convenient to do both. Accordingly, the analytic formulas provided below. Denote

$$\tilde{\tilde{y}} \equiv \tilde{y}\left[(1-\sigma)\ \exp(\nu x)\tilde{k}^{-\nu}+\sigma\tilde{l}^{-\nu}\right]^{-1},$$

where variables with a tilde are defined in (3.1) below and $\tilde{y}$ appears in (3.6a). The link to the starred variables is given in (3.2). Then,

$$r_2 = -\frac{\exp(u)}{\tilde{c}}\exp(-x)$$

$$r_1 = \frac{-r_2}{\beta}$$

$$r_4 = -\frac{\exp(u)}{\tilde{c}}$$

$$r_3 = \frac{-r_4}{\beta}$$

$$r_5 = 0$$

$$r_6 = 0$$

$$r_7 = -\frac{\exp(u)}{\tilde{c}}\left\{\theta(1-\sigma)\ \exp(\nu x)\tilde{k}^{-\nu}\tilde{\tilde{y}}+\frac{1-\delta}{n}\exp(-2x)\tilde{k}\right\}$$

$$r_8 = \exp(u)\ \ln\tilde{c}.$$

$$r_9 = \frac{\exp(u)}{\tilde{c}}\left\{-\theta\tilde{y}+\exp(-x)\tilde{k}\left[1-\frac{1-\delta}{n}\exp(-x)\right]-\exp(-x)\frac{1}{n}\right\}.$$

The expressions for r_1, r_3, and r_5 exploit the steady state first order necessary conditions.

Next, turn to the second derivatives.

$$r_{11} = -r_1^2 \exp(-u) - \{\theta(1-\sigma) \exp(\nu x)\tilde{k}^{-(\nu+2)} \exp(u)/\tilde{c}\}$$

$$\times \{(\nu+1)\tilde{\tilde{y}}-(\theta+\nu)(1-\sigma) \exp(\nu x)\tilde{\tilde{\tilde{y}}}\tilde{k}^{-\nu}\}$$

where,

$$\tilde{\tilde{\tilde{y}}} = \tilde{\tilde{y}}[(1-\sigma) \exp(\nu x)\tilde{k}^{-\nu}+\sigma\tilde{l}^{-\nu}]^{-1}.$$

Also,

$$r_{12} = -r_1 r_2 \exp(-u)$$

$$r_{13} = -r_1 r_3 \exp(-u) + \frac{\exp(u)}{\tilde{c}} \theta(1-\sigma) \exp(\nu x)(\theta+\nu)\sigma\tilde{\tilde{\tilde{y}}}(\tilde{k}\tilde{l})^{-(\nu+1)}$$

$$r_{14} = -r_1 r_4 \exp(-u)$$

$$r_{15} = -r_1 \gamma \exp(-u) + \frac{\exp(u)}{\tilde{c}} \theta(1-\sigma) \exp(\nu x)(1-\theta)\tilde{\tilde{y}}\tilde{k}^{-(\nu+1)}$$

$$r_{16} = 0$$

$$r_{17} = -r_1 r_7 \exp(-u) + \frac{\exp(u)}{\tilde{c}} \{\nu\theta(1-\sigma) \exp(\nu x)\tilde{\tilde{y}}\tilde{k}^{-(\nu+1)}-(\theta+\nu)\theta(1-\sigma)^2$$

$$\exp(2\nu x)\tilde{k}^{-(2\nu+1)}\tilde{\tilde{\tilde{y}}} - \frac{1-\delta}{n} \exp(-2x)\}$$

$$r_{18} = r_1$$

$$r_{19} = -r_1 r_9 \exp(-u) - \frac{\exp(u)}{\tilde{c}} \{\theta^2(1-\sigma) \exp(\nu x)\tilde{k}^{-(\nu+1)}\tilde{\tilde{y}}+ \frac{1-\delta}{n} \exp(-2x)\}$$

$$r_{22} = -r_2^2 \exp(-u), \quad r_{23} = -r_2 r_3 \exp(-u)$$

$$r_{24} = -r_2 r_4 \exp(-u), \quad r_{25} = -r_2 \gamma \exp(-u)$$

$$r_{26} = 0$$

$$r_{27} = -r_2 r_7 \exp(-u)$$

$$r_{28} = r_2$$

$$r_{29} = -r_2 r_9 \exp(-u) + \frac{\exp(u)}{\tilde{c}} \exp(-x)$$

$$r_{33} = -r_3^2 \exp(-u) + \theta\sigma \frac{\exp(u)}{\tilde{c}} \{-(\nu+1)\tilde{\tilde{y}}\tilde{i}^{-(\nu+2)} + (\theta+\nu)\sigma\tilde{\tilde{\tilde{y}}}\tilde{i}^{-2(\nu+1)}\}$$

$$r_{34} = -r_3 r_4 \exp(-u)$$

$$r_{35} = -r_3 \gamma \exp(-u) + \frac{\exp(u)}{\tilde{c}} \theta\sigma\tilde{i}^{-(\nu+1)}(1-\theta)\tilde{\tilde{y}}/h$$

$$r_{36} = 0$$

$$r_{37} = -r_3 r_7 \exp(-u) - \frac{\exp(u)}{\tilde{c}} \theta\sigma (\theta+\nu)(1-\sigma)\tilde{i}^{-(\nu+1)} \exp(\nu x)\tilde{k}^{-\nu}\tilde{\tilde{\tilde{y}}}$$

$$r_{38} = r_3$$

$$r_{39} = -r_3 r_9 \exp(-u) + \frac{\exp(u)}{\tilde{c}} \{-\theta^2\sigma\tilde{i}^{-(\nu+1)} - \exp(-x)/n\}$$

$$r_{44} = -r_4^2 \exp(-u)$$

$$r_{45} = -r_4 \gamma \exp(-u)$$

$$r_{46} = 0$$

$$r_{47} = -r_4 r_7 \exp(-u)$$

$$r_{48} = r_4$$

$$r_{49} = -r_4 r_9 \exp(-u)$$

$$r_{55} = -\gamma^2 \exp(-u) - \theta(1-\theta) \frac{\exp(u)}{\tilde{c}} \tilde{y}/h^2$$

$$r_{56} = 0$$

$$r_{57} = \frac{\gamma}{\tilde{c}} \{\tilde{\tilde{y}}\theta(1-\sigma)\ \exp(\nu x)\tilde{k}^{-\nu} + \frac{1-\delta}{n} \exp(-2x)\tilde{k}\}$$
$$- \frac{\exp(u)}{\tilde{c}} (1-\theta)\tilde{\tilde{y}}\theta(1-\sigma)\ \exp(\nu x)\tilde{k}^{-\nu}/h$$

$$r_{58} = \gamma$$

$$r_{59} = -\gamma r_9 \exp(-u) - \frac{\exp(u)}{\tilde{c}} (1-\theta)\theta\tilde{y}/h$$

$$r_{66} = \ldots = r_{69} = 0$$

$$r_{77} = -r_7 r_7 \exp(-u) + \frac{\exp(u)}{\tilde{c}} \{\theta(1-\sigma)\ \exp(\nu x)\tilde{k}^{-\nu}\tilde{\tilde{y}}(\theta+\nu)(1-\sigma)\ \exp(\nu x)\tilde{k}^{-\nu}$$
$$+ \frac{1-\delta}{n} \exp(-2x)\tilde{k}\}$$

$$r_{78} = r_7$$

$$r_{79} = -r_7 r_9 \exp(-u) + \frac{\exp(u)}{\tilde{c}} \{\theta^2(1-\sigma)\ \exp(\nu x)\tilde{k}^{-\nu}\tilde{\tilde{y}} + \frac{1+\delta}{n} \exp(-2x)\tilde{k}\}$$

$$r_{88} = r_8$$

$$r_{89} = r_9$$

$$r_{99} = -r_9^2 \exp(-u) + \frac{\exp(u)}{\tilde{c}} \{\theta^2\tilde{y} - \exp(-x)\tilde{k} + \frac{1-\delta}{n} \exp(-2x)\tilde{k} + \exp(-x)\tilde{I}/n\}.$$

3. Derivation of the Steady State Formulas in Section 3.b of the Paper

In section 3.b I display formulas for the logarithm of the steady state of k_t/z_{t-1}, i_t/z_t, and h_t. These are derived in this appendix. It is convenient to first derive the formulas for the levels, and to convert to logs at the last step. Define

(3.1) $\tilde{y}_t = y_t/z_t$, $\tilde{c}_t = c_t/z_t$, $\tilde{k}_t = k_t/z_{t-1}$, $\tilde{I}_t = i_t/z_t$, and let $\tilde{y}$, $\tilde{c}$, $\tilde{k}$, $\tilde{i}$, h denote the steady state values of $\tilde{c}_t$, $\tilde{k}_t$, $\tilde{i}_t$, and h_t, respectively. In addition, let w denote the steady state value of w_t, so that, trivially, $w = (I-A)^{-1}a = (u,x)^T$. The link between (3.1) and the starred variables in the text is given by

$$(3.2) \qquad \tilde{c}_t = \exp(c_t^*),\ \tilde{i}_t = \exp(i_t^*),$$

$$\tilde{k}_t = \exp(k_t^*),\ \tilde{y}_t = \exp(y_t^*),\ h_t = \exp(h_t^*).$$

In terms of these variables, the planning problem is to maximize

$$(3.3) \qquad E_0\sum_{t=0}^{\infty}\beta^t\tilde{r}(\tilde{k}_{t-1},\tilde{k}_t,\tilde{i}_{t-1},\tilde{i}_t,h_t,w_{t-1},w_t),$$

subject to the information structure and the initial conditions. Here,

$$(3.4) \qquad \tilde{r}(\tilde{k}_{t-1},\tilde{k}_t,\tilde{i}_{t-1},\tilde{i}_t,h_t,w_{t-1},w_t) \equiv r\big(\log(\tilde{k}_{t-1}),\log(\tilde{k}_t),\log(\tilde{i}_{t-1}),$$

$$\log(\tilde{i}_t),\log(h_t),w_{t-1},w_t\big)$$

and r is defined before (3.5) in the paper.

The first order necessary conditions satisfied by $\tilde{k}$, $\tilde{i}$, h are, respectively,

$$\tilde{r}_2(\tilde{k},\tilde{k},\tilde{i},\tilde{i},h,w,w) + \beta\tilde{r}_1(\tilde{k},\tilde{k},\tilde{i},\tilde{i},h,w,w) = 0$$

$$\tilde{r}_4(\tilde{k},\tilde{k},\tilde{i},\tilde{i},h,w,w) + \beta\tilde{r}_3(\tilde{k},\tilde{k},\tilde{i},\tilde{i},h,w,w) = 0$$

$$\tilde{r}_5(\tilde{k},\tilde{k},\tilde{i},\tilde{i},h,w,w) = 0,$$

where $\tilde{r}_j$ denotes the derivative of $\tilde{r}$ with respect to its j-th argument. The first relation states that the utility cost of increasing the current (detrended) stock of capital must equal the discounted utility benefit from the resulting increase in (detrended) consumption in the next period. The second and third relations have analogous interpretations for the stock of inventories and hours, respectively. For my particular parametric example, the above formulas are:

$$\text{(3.5a)} \qquad \exp(-x) = \beta\{\theta(1-\sigma)\ \exp(\nu x)\tilde{k}^{-(\nu+1)}\tilde{y}[(1-\sigma)\ \exp(\nu x)\tilde{k}^{-\nu}+\sigma\tilde{I}^{-\nu}]^{-1} + \frac{1-\delta}{n}\exp(-2x)\}$$

$$\text{(3.5b)} \qquad 1 = \beta\{\theta\sigma\tilde{I}^{-(\nu+1)}\tilde{y}[(1-\sigma)\ \exp(\nu x)\tilde{k}^{-\nu}+\sigma\tilde{I}^{-\nu}]^{-1} + \exp(-x)\ \frac{1}{n}\}$$

$$\text{(3.5c)} \qquad \frac{\exp(u)}{\tilde{c}}\ (1-\theta)\ \frac{\tilde{y}}{h} = \gamma.$$

Here,

$$\text{(3.6a)} \qquad \tilde{y} = n^{-\theta}h^{(1-\theta)}\ \exp(-\theta x)[(1-\sigma)\ \exp(\nu x)\tilde{k}^{-\nu}+\sigma\tilde{I}^{-\nu}]^{-(\theta/\nu)}.$$

Also, the steady state resource constraint yields

$$\text{(3.6b)} \qquad \tilde{c} = \tilde{y} - \exp(-x)\tilde{k} + \frac{1-\delta}{n}\exp(-2x)\tilde{k} - \tilde{I} + \exp(-x)\ \frac{1}{n}\ \tilde{I}.$$

Equations (3.5a) - (3.6b) represent five equations in the five unknowns, $\tilde{y}$, h, $\tilde{I}$, $\tilde{k}$, $\tilde{c}$. We proceed now to obtain their unique solution.

First, note from (3.5b) that

$$\tilde{y}[(1-\sigma)\ \exp(\nu x)\tilde{k}^{-\nu}-\sigma\tilde{I}^{-\nu}]^{-1} = \frac{1 - \beta\ \exp(-x)/n}{\beta\theta\sigma\tilde{I}^{-(\nu+1)}}.$$

Substituting this into (3.5a), get

$$\exp(-x) = \beta\theta(1-\sigma)\ \exp(\nu x)\tilde{k}^{-(\nu+1)}[1-\beta\ \exp(-x)/n]/[\beta\theta\sigma\tilde{I}^{-(\nu+1)}]$$
$$+ \beta\ \frac{1+\delta}{n}\exp(-2x) = (\frac{1-\sigma}{\sigma})\ \exp(\nu x)[1-\beta\ \exp(-x)/n](\frac{\tilde{k}}{\tilde{I}})^{-(\nu+1)}$$
$$+ \beta\ \frac{1-\delta}{n}\exp(-2x).$$

Conclude,

$$\frac{\tilde{I}}{\tilde{k}} = \left\{\frac{\sigma[1-\beta(1-\delta)\ \exp(-x)/n]}{(1-\sigma)\ \exp[(\nu+1)x][1-\beta\ \exp(-x)/n]}\right\}^{(\frac{1}{\nu+1})}$$

or

$$(3.7a)\qquad \tilde{I} = \lambda\tilde{k}$$

where,

$$(3.7b)\qquad \lambda = \left\{\frac{\sigma[1-\beta(1-\delta)\ \exp(-x)/n]}{(1-\sigma)\ \exp[(\nu+1)x][1-\beta\ \exp(-x)/n]}\right\}^{(\frac{1}{1+\nu})}.$$

Substitute (3.7a) and (3.6a) into (3.5a) to get

$$(3.8a)\qquad \tilde{k} = \psi h$$

where

(3.8b)

$$\psi = \left\{\frac{n^{\theta}\ \exp[(\theta-\nu-1)x][1-\beta(1-\delta)\ \exp(-x)/n][(1-\sigma)\ \exp(\nu x)+\sigma\lambda^{-\nu}]^{(\frac{\theta+\nu}{\nu})}}{\beta\theta(1-\sigma)}\right\}^{(\frac{-1}{1-\theta})}.$$

Next, I use (3.6a), (3.6b), (3.7a), (3.8a) to transform (3.5c) into one equation in h. Begin by substituting (3.7a) and (3.8a) into (3.6a) and (3.6b):

$$(3.9)\qquad \tilde{y} = hn^{-\theta}\ \exp(-\theta x)\psi^{\theta}[(1-\sigma)\ \exp(\nu x)+\sigma\lambda^{-\nu}]^{-(\theta/\nu)} = h\alpha_1$$

$$(3.10)\qquad \tilde{c} = \tilde{y} - h\ \{\psi\ \exp(-x)[1-(1-\delta)\ \exp(-x)/n]+\lambda\psi[1-\ \exp(-x)\ \tfrac{1}{n}]\} = h\alpha_2.$$

Here,

$$\alpha_1 = n^{-\theta}\ \exp(-\theta x)\psi^{\theta}[(1-\sigma)\ \exp(\nu x)+\sigma\lambda^{-\nu}]^{-\theta/\nu}$$

$$\alpha_2 = \alpha_1 - \{\psi \exp(-x)[1-(1-\delta) \exp(-x)/n]+\lambda\psi[1- \exp(-x)/n]\}.$$

Substituting these into (3.5c), get

$$(3.11) \qquad h = \frac{\exp(u)}{\gamma} (1-\theta)(\alpha_1/\alpha_2).$$

Equations (3.7a), (3.8a), and (3.11) are the steady state equations sought.

4. Data

Following is a discussion of the data used in this project.

Quality Adjusted, Working Age Population

Data on the total male and female working age population were obtained from the Chase Econometrics U.S. Macroeconomic data base. The working aged population was defined as males and females aged 15 to 64. The Chase mnemonics for these data are ANPTMT1519, ANPTMT2024, ..., ANPTMT6064, and ANPTFT1519, ANPTFT2024, ..., ANPTFT6064, respectively, and they were most recently revised on April 28, 1986. The data are available on an annual basis, and represent estimates of the population on July 1.

The model of this paper abstracts from the effects of changes in human capital on labor productivity. However, the human capital of the average worker in the post-war period has not been constant. In an attempt to adjust for this, I obtained a quality adjusted working age population by weighting each age-sex group by its average wage in the 1970's. The weights, which were standardized on males aged 35 to 44, were taken from Hansen (1984), and are as follows:

Table A1

Ages	Males	Females
15-19	.44	.4
20-24	.61	.51
25-34	.86	.64
35-44	1.0	.63
45-54	1.01	.62
55-64	.95	.6

The gross growth rate in the quality adjusted (GWEIT) and unadjusted (GUNWT) working aged populations are graphed in Figure 1A. The results there show that adjusting the data along age-sex lines does not have a substantial effect on the numbers. Darby (1984) argues that the data ought to be adjusted for education levels and immigration flows. This further adjustment may be worth exploring, however, I have not done so.

Quarterly observations on the quality adjusted working age population were obtained by log-linearly interpolating the annual data. The calculations were carried out treating the annual observations as third quarter observations.

Several features of the data stand out. First, as is plain from Figure 1A, they do not satisfy the constant growth assumption in the text. This is confirmed by the numbers in Table 2A.

Table 2A:

Percent Annual Growth Rate, Working Age Population

	Quality Adjusted	Not Adjusted
1952-1961	.85 (.07)	.9 (.19)
1970-1984	1.6 (.19)	1.5 (.35)
1949-1984	1.3 (.36)	1.3 (.43)

Numbers in parenthesis in Table 2A are standard deviations, in percent terms. The growth rate of the quality adjusted working age population approximately doubled in the 1970's and 1980's over what it was in the 1950's. Interestingly, the results are basically the same for the unadjusted working aged population.

A second important feature of the working age population is that they behave very differently from data on the population as a whole. Data on the total population, including armed forces overseas, were obtained from the Chase Econometrics database (mnemonic NPT). From the period 1952 to 1961, this data display an average annual growth rate of 1.7 percent, with standard error .07 percent. For the period 1970 to 1984, the average growth rate was 1 percent with standard error 0.1 percent. Thus, the pattern of growth in the working age population is opposite to that of the population as a whole. This probably reflects the large number of births shortly after the war, which showed up in the total population immediately, but only with a lag in the working age population. Because of this, the time series behavior of economic variables in per capita terms are sensitive to the choice of population data used.

A third feature of this population data is that the growth in total population exhibits substantial seasonality, with growth being especially high in the first few months of the year. Obviously my interpolated quality adjusted working age population data do not exhibit such seasonality, although the actual working age population probably does. The absence of seasonality in my working age population data is consistent with the fact that all other data used in this project have been seasonally adjusted.

Capital Stock and Investment

Investment for the purpose of this project is defined as real consumer purchases of durable goods (ECD) plus real gross private fixed domestic investment (IFIXED), plus real government (federal, state, and local) investment, including investment in the military (IGINVEST). ECD and IFIXED are as reported in Table 1.2 of the Survey of Current Business (SCB). Annual observations on IGINVEST were provided to me by John Musgrave of the Bureau of Economic Analysis. The IGINVEST data are a revised and updated version of the government investment data discussed in Musgrave [1980]. Quarterly observations in IGINVEST were obtained using the interpolation by related series method of Chow and Lin (1972). The related series used for this purpose were ECD, IGD82, a constant and a linear trend. (IGD82 is gross private domestic investment, as reported in Table 1.2 of SCB.)

The aggregate investment data were converted to per capita terms by dividing by the quality adjusted working age population.

Annual, end of year capital stock data were obtained from the January, 1986 SCB, Tables 4, 8, 12, 16, 20, found on pages 59-75. These data were used to obtain a time series on the 1982 dollar value of the net stock of capital. The data are the sum of fixed nonresidential capital (private, federal, state and local), plus the stock of durable goods held by consumers, plus the stock of government and privately held residential capital. For further details about this data, the reader is referred to the data source.

Some details about the composition of the capital stock are of interest. First, the average value of the capital to quarterly GNP ratio in the period 1955 QIII to 1984 QI is 12.8, with a standard deviation of .5. (Here, GNP is defined as GNP according to National Income account standards, plus the services of consumer durables, minus net exports.) At the end of

1984, the aggregate net stock of capital was $9,799 billion, in 1982 dollars. Of this, 35 percent was private equipment and structures, consumer durables were 13 percent, public and private residential capital was 32 percent and government equipment and structures was 20 percent.

Since the capital data in the SCB are annual, they had to be converted to quarterly observations. Quarterly observations on consumer durables and private equipment and structures were obtained from the MPS model data base. (This is documented in Brayton and Manskopf [1985].) Data on the private stock of residential capital were also obtained from the MPS data base. These data, together with a constant and linear trend were used to interpolate the annual public and private stock of residential capital data in the SCB. (The method of interpolation by related series due to Fernandez [1981] was used for this.) A quarterly series on government equipment and structures was obtained by log-linearly interpolating the annual data taken from the SCB.

Finally, a quarterly per capita data series on the aggregate stock of capital was obtained by adding the individual components and dividing the result by the quality adjusted, working age population.

The depreciation rate on capital, δ, plays an important role in this paper for two reasons. First, it is a parameter of the model so that the value it is assigned has implications for the average capital to output ratio and other endogenous quantities. Second, in order to deduce my model's implications for capital investment, I have to quasi first difference the capital stock series that it generates, using some value for δ.

Based on my examination of the capital stock data, I decided to set $\delta = .018$, which is 7.4 percent annually. This is lower than the numbers used by other researchers. (For example, Kydland and Prescott [1982] assume 10 percent annual depreciation.) The reason I did this was that key time series properties of the actual investment data coincide with investment data derived from my capital stock series using $\delta = 0.018$. This is not the case when a 10 percent annual depreciation rate is assumed.

The regression of per capita capital (k_t) minus per capita gross fixed investment (dk_t) on k_{t-1} produced a coefficient of 0.9787. The sample period was 1955 QIV to 1984 QI. In terms of the model in the text, this regression coefficient is to be interpreted as a measure of $(1-\delta)/n$, where n is the quarterly gross growth rate of the working age population and δ is the quarterly depreciation rate. With $n = (1.013)^{.25}$, this implies $\delta = 0.018$, an annual depreciation rate of about 7.4 percent.

Unfortunately, $k_t - .9787k_{t-1}$ and dk_t differ by a substantial amount. The average value of $100|k_t - .9787k_{t-1} - dk_t|/|dk_t|$ is five percent for the period 1955,4 - 1984,1. (Here, $|\cdot|$ denotes the absolute value operator.) Moreover, the discrepancy, $k_t - .9787k_{t-1} - dk_t$ is highly serially correlated throughout the sample, being strictly positive before 1970, strictly negative thereafter, and close to zero on average.

These results are not consistent with my model formulation, although there may be reason to believe that the consequences of this misspecification are not serious. This is because $k_t - .9787k_{t-1}$ shares several key time series properties of dk_t. First, both are on average 27 percent of gross output. Also, the mean of $100|(k_t-.9787k_{t-1})-(k_{t-1}-.9787k_{t-2})|/y(t)$ and $100|dk_t-dk_{t-1}|/y(t)$ are roughly the same. The former is 0.46 with standard deviation 0.40, while the latter is 0.47 with standard deviation 0.39. If $\delta = 0.025$ is used, then the derived investment series is shifted up by a large 496 dollars per person on average. This is just the product of $(0.025-0.018)/n$ and the average value of the stock of capital, which is large relative to investment. As a result of this, the average share of gross output of this investment series is 34 percent, substantially higher than the actual, 27 percent figure. Because of this, I set $\delta = 0.018$ in this study and did not use the more conventional $\delta = 0.025$.

Inventories

Inventory investment was defined as the change in farm and nonfarm inventories in 1982 dollars as reported in Table 5.9 of SCB. The stock of farm and nonfarm inventories is as reported in Table 5.11 of SCB.

Quality Adjusted Hours Worked

Time series for hours worked for the period 1955 Q3 to 1984 Q1 were provided to me by Gary Hansen. The underlying data were obtained from the Current Population Survey, which is a survey of households. The data were then aggregated by age-sex groups using the weights reported in Table 1A. For further details about this data and the manner in which they were constructed, see Hansen (1984).

As I noted earlier, Darby [1984] argues that data ought to be further adjusted to reflect changes in education levels and immigration flows. Darby provides an annual hours series adjusted in this way for the period 1900-1979 (his mnemonic is QATHWP). The gross rate of change in this data (I call it GDARBY) and in Hansen's quality adjusted hours series (GHANSEN) appear in Figure 2A. The difference between these two series is not great, suggesting that my analysis is probably not sensitive to adjustments for immigration and education.

I obtained a per capita hours series by dividing quality adjusted hours worked by the quality adjusted working age population. These data are graphed in Figure 3A. My model implies a per capital hours series that fluctuates about a constant mean. The data, in fact, show very slight evidence of an increase in hours worked per capita in the post war period. Average growth in the per capita hours series is 0.16 percent annually. On the other hand, the standard deviation is an enormous 6 percent.

Consumption

The measure of consumption I used is consumption of nondurables plus consumption of services plus the imputed rental value of the stock of consumer durables, plus government consumption. All these components except the last two were taken from SCB. A measure of the imputed rental value of consumer durables was obtained from the data base documented in Brayton and Mauskopf [1985]. Government consumption is government purchases of goods and services minus IGINVEST.

Per capita consumption was obtained by dividing by the quality adjusted, working aged population.

GROSS GROWTH RATE
POPULATION AGED 15 — 65 YEARS

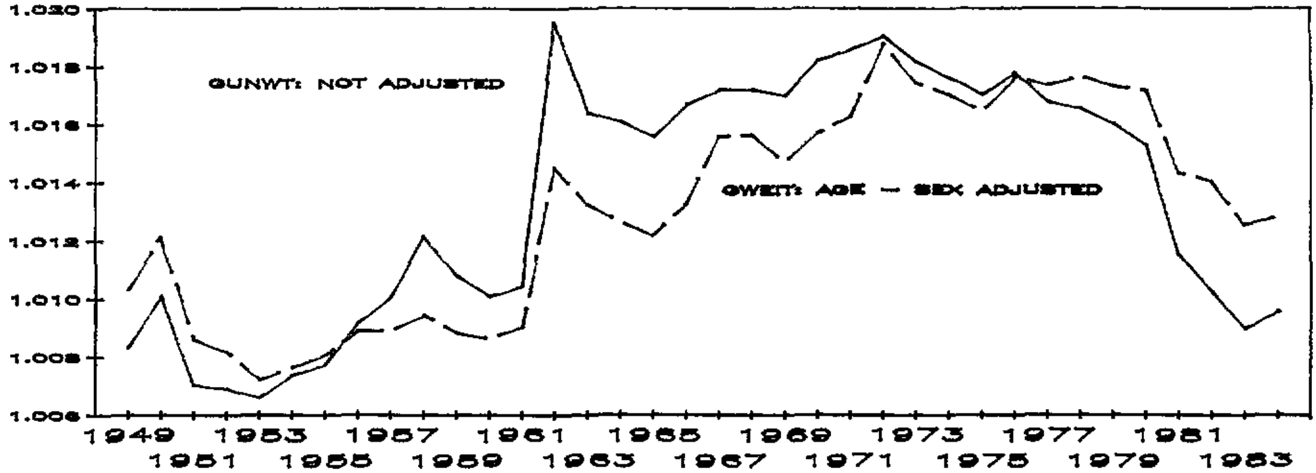

FIGURE 1A

GROSS GROWTH RATE
QUALITY ADJUSTED AGGREGATE HOURS

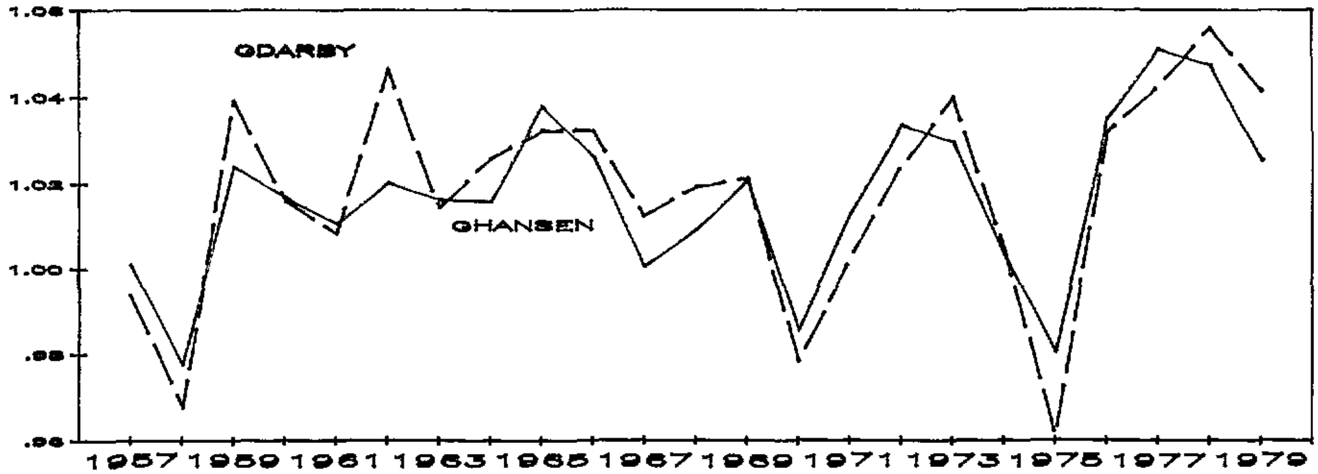

FIGURE 2A

RATIO
QUALITY ADJUSTED HOURS TO QUALITY ADJUSTED WORKING AGED POPULATION

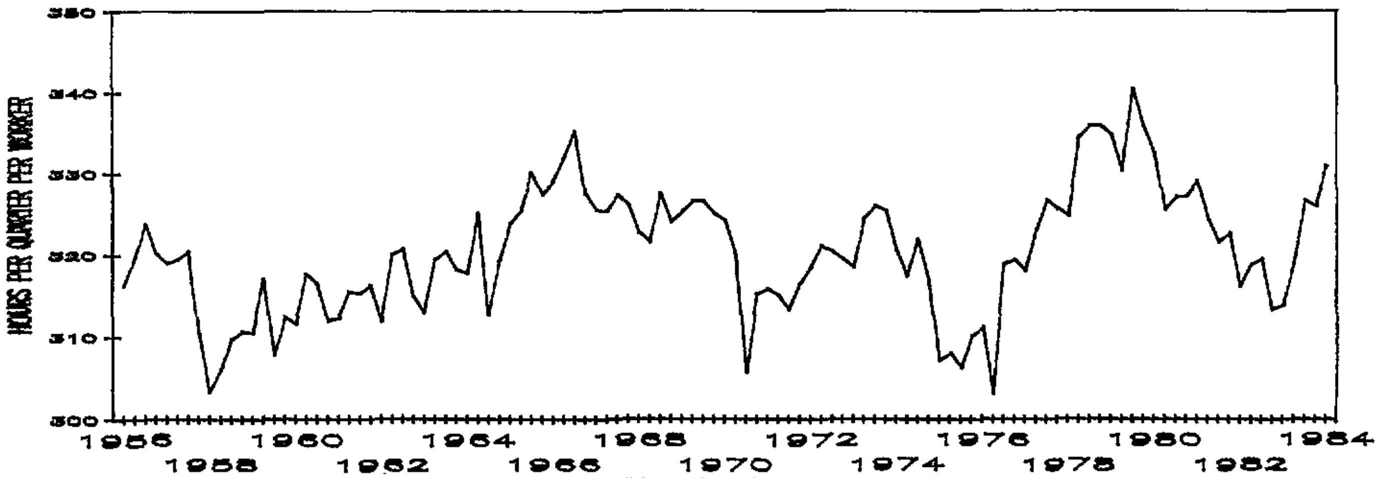

FIGURE 3A