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TIME TO BUILD AND AGGREGATE FLUCTUATIONS: SOME NEW EVIDENCE

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ABSTRACT

This paper presents maximum likelihood estimates of a real business cycle model very similar to one Kydland and Prescott [1982] suggested. The results of the paper conflict with Kydland and Prescott's. The model leaves unexplained much of the variance of two key investment series, namely, structures and equipment. Also, much of the variation in the differences of per capita hours can be generated assuming that past leisure choices do not affect current utility.

1. Introduction

In a paper that has received much recent attention, Kydland and Prescott [1982] presented a competitive equilibrium model of cyclical fluctuations.[1/] Using the one-sector optimal growth model to construct a prototype competitive economy, they argued that postwar U.S. business cycles could be explained in terms of the dynamic response of the aggregate economy to persistent technology shocks. To provide a more complicated propagation mechanism for such shocks, they modified the optimal growth model by introducing a time-to-build feature in investment and by allowing leisure to be a durable good. An informal empirical analysis--which involved matching a small set of moments generated by the model with their sample counterparts for the postwar U.S. economy--led these authors to conclude that the model's "fit is very good, particularly in light of the model's simplicity" (p. 1363). Among other results, they reported that "the model is consistent with the large (percentage) variability in investment and low variability in consumption and their high correlations with real output" (p. 1364). With respect to specific features of the model, they argued that "the dependence of current utility on past leisure choices . . . is crucial in making [the model] consistent with the observation that cyclical employment fluctuates substantially more than productivity does" (p. 1367).

This paper presents maximum likelihood estimates of a model that has the main features of Kydland and Prescott's model. It uses postwar U.S. data on the differences of per capita values of aggregate output, total hours worked, and investment expenditures for two types of investment, namely, in structures and in equipment. Contrary to Kydland and Prescott's assertion, I find that the model fails most drastically in its ability to explain the variability of the two investment series. The source of this failure is not the time-to-build feature, as might be expected. Instead, it is a more fundamental feature of the model: the existence of a well-behaved neoclassical production technology describing the relationship between aggregate output and the inputs of labor and the two types of capital. More precisely, I estimate the parameter which determines the share of labor in aggregate output--and, hence, the elasticity of output with respect to labor--to be unity. With a constant returns to scale production function, such a finding implies that the composite capital good involving the stock of structures and equipment is, in effect, driven out of the aggregate production function. As the model is specified, the behavior of the two investment series is directly linked to the behavior of the two capital series, but little role emerges for capital with a unitary labor share parameter. Thus, the model can generate only a fraction of the variability in the two investment series which the actual data or other simpler specifications display.

I do find, however, that the model explains quite well the behavior of the hours series under a time-separable specification of preferences. But the evidence for the dependence of current utility on past leisure choices is, at best, mixed. When the effect of past leisure choices on current leisure services is parametrized by an infinite distributed lag and the parameter γ, which determines the curvature of the single-period utility function in a composite consumption good, is freely estimated, the durability of leisure turns out to be unimportant for explaining the variation in per capita hours or its covariation with per capita output and investment. Some evidence emerges for a positive effect from single lagged values of leisure to current utility when the service flow technology determining leisure services l_t^* is parametrized as $l_t^* = l_t + \beta_1 l_{t-1}$ and γ is fixed at unity. However, what is perhaps an equally interesting result is that my estimates of the preference parameters γ and b, which jointly determine the curvature of the single-period utility function with respect to consumption c_t and leisure services l_t^*, are consistent with an infinite intertemporal substitution elasticity for the composite good involving consumption and leisure services and with some type of increasing returns with respect to per capita leisure hours in the representative consumer's preferences.

Kydland and Prescott's approach and mine differ in several ways. Some concern the specification of the model and the so-called detrending procedure. Such differences are minor, however, and were motivated on empirical grounds. Our two studies also use somewhat different data sets. My data set is slightly smaller than Kydland and Prescott's--it does not include series on consumption expenditures, aggregate inventories, capital stocks, or productivity--but it does contain a sufficiently diverse set of series whose behavior can be used to investigate the important features of Kydland and Prescott's model. The most important difference between the two papers concerns the estimation procedure: Kydland and Prescott calibrated a singular stochastic model using a small set of sample moments--the variances of the detrended series, their correlation with detrended output, and five autocorrelations of detrended output. By contrast, I derive, as an econometric specification, a restricted index model in which the innovation to the technology shock appears as the common latent factor, while serially uncorrelated measurement errors constitute specific disturbances. A frequency domain approximation to the exact likelihood function for the vector of observable series then delivers estimates of a major subset of the parameters characterizing preferences and technology.

One way to summarize my results, therefore, is to note that, when a major subset of the unknown parameters is freely estimated, using full sample information, many of Kydland and Prescott's conclusions disappear, and they disappear in ways difficult to predict on a priori grounds. This paper may be viewed in another way, however. It provides an empirical investigation of a real business cycle model, which is similar to Kydland and Prescott's, but which also allows for additional specifications describing the durability of leisure, accounts for potential differences in the behavior of the stock of structures versus the stock of equipment, and incorporates nonstationary behavior for the latent technology shock.

2. The Model

2.1 Specification

Here I briefly describe the model estimated in this paper and, where appropriate, point to the differences with the Kydland-Prescott specification. I start with preferences: the utility functional of the representative consumer at time zero is

$$(2.1) \qquad U = \sum_{t=0}^{\infty} \beta^t u(c_t, l^*_t) = \sum_{t=0}^{\infty} \beta^t \frac{1}{\gamma} [c_t^b l_t^{*1-b}]^\gamma$$

where β is the discount factor, c_t the consumption of the single good, and l^*_t the service flow derived from current and past leisure choices. The parameter b, which represents the share of consumption in current utility, is constrained to be strictly between zero and one, while the risk aversion parameter γ is restricted as $0 \neq \gamma < 1$. A value of $\gamma = 1$ implies risk neutrality in the consumption of the composite good $c_t^b l_t^{*1-b}$. Tastes are assumed to be constant and uninfluenced by exogenous random shocks.

The durability of leisure is modeled in two simple ways. According to one, current and all past values of leisure affect current utility with geometrically declining weights, that is,

$$(2.2) \qquad l_t^* = (1-\eta)\sum_{i=0}^{\infty} \eta^i l_{t-i}$$

where $0 < \eta < 1$ and l_t denotes current leisure. Normalizing the total available time as one, letting $n_t = 1 - l_t$ denote current labor supply, and defining the stock variable $a_t = \sum_{i=1}^{\infty} \eta^{i-1} n_{t-i}$ yields the representation

$$(2.3a) \qquad l_t^* = 1 - (1-\eta)n_t - \eta(1-\eta)a_t$$

$$(2.3b) \qquad a_{t+1} = \eta a_t + n_t.$$

In this specification, the parameter η measures the rate at which the effect of past leisure choices on current utility decreases over time, as well as determining the degree to which leisure in different periods can be substituted for each other. Kydland and Prescott assumed instead a specification for l_t^* which allows past leisure choices to affect current utility, even if the effect of leisure consumed in the past decays rapidly. More precisely, they assumed that

$$(2.4a) \qquad l_t^* = 1 - \alpha_0 n_t - \eta(1-\alpha_0)a_t$$

$$(2.4b) \qquad a_{t+1} = (1-\eta)a_t + n_t$$

with $0 < \alpha_0 < 1$, $0 < \eta < 1$, and $a_t = \sum_{i=1}^{\infty}(1-\eta)^{i-1} n_{t-i}$. However, (2.3) and (2.4) share the feature that whenever η is different from zero (or α_0 different from one), only a fraction of current leisure hours provides current utility.

A second parametrization for l_t^* estimated here but not by Kydland and Prescott allows past values of leisure to have possible negative effects on current utility, that is,

$$(2.5) \qquad l_t^* = l_t + \beta_1 l_{t-1}$$

with $\beta_1 > -1$.

Aggregate output is produced according to the CES production function, with the inputs of labor n_t and two types of capital k_{1t} and k_{2t}, as

$$(2.6) \qquad Q_t = f(\lambda_t, n_t, k_{1t}, k_{2t}) = \lambda_t n_t^{\theta}[(1-\sigma)k_{1t}^{-\nu} + \sigma k_{2t}^{-\nu}]^{\frac{-(1-\theta)}{\nu}}$$

with $0 < \theta < 1$, $0 < \sigma < 1$, $-1 < \nu < \infty$, and λ_t a random shock to technology. The parameter ν determines the elasticity of substitution between the two types of capital, and θ measures the share of labor in aggregate output. When taking the model to data, I identify k_{1t} and k_{2t} with the stocks of structures and equipment, respectively. By contrast, Kydland and Prescott consider a single type of capital good but assign a productive role to inventories. Hence, aggregate inventories appear as the third input in their specification of the production function.

The technology shock λ_t evolves according to the law of motion

$$(2.7) \qquad \lambda_t = \lambda_{1t} + \lambda_{2t} = \lambda_{1t-1} + \varepsilon_t + \bar{\lambda}$$

where $\bar{\lambda} > 0$ and $\{\varepsilon_t\}_{t=0}^{\infty}$ are identically and independently distributed with mean zero and variance σ_ε^2. Hence, $\lambda_{2t} = \bar{\lambda}$ for all t while λ_{1t} behaves as a simple random walk. According to this representation, the technology shock also behaves as a simple random walk, the differences $\lambda_t - \lambda_{t-1}$ moving randomly around a mean of zero. I adopt the representation described by equation (2.7) for two reasons. Initial estimates of the model using Kydland and Prescott's specification of a stationary first-order autoregressive process for the technology shock showed that the transitory components to productivity were unimportant for explaining the data while the autoregressive parameter was driven to unity.[2/] The reason for assuming that the technology process evolves without a drift term is to ensure that the approximation procedure used to derive an econometric model has a consistent interpretation. However, I will show that the model estimated here is, with a small modification, also consistent with another specification for the technology shock, one which assumes that λ_t is a random walk with a drift equal to $\overline{\lambda}$.

Different assumptions characterize investment in the two types of capital. Investment in equipment proceeds under standard neoclassical assumptions. The implied law of motion is

$$k_{2t+1} = (1-\delta_2)k_{2t} + i_{2t}. \tag{2.8}$$

Investment in structures is characterized by the time-to-build requirement:

$$s_{j,t+1} = s_{j+1,t} \qquad \text{for } j = 1, \ldots, J-1 \tag{2.9}$$

$$k_{1t+1} = (1-\delta_1)k_{1t} + s_{1t} \tag{2.10}$$

$$i_{1t} = \sum_{j=1}^{J} \phi_j s_{jt}. \tag{2.11}$$

In these equations, s_{jt} denotes the number of investment projects which are j periods from completion and s_{Jt} the investment projects initiated in period t. The fraction of resources expended on a project j periods from completion is fixed and given by ϕ_j for j = 1, ..., J, with $\phi_j > 0$ and $\sum_{j=1}^{J} \phi_j = 1$. Hence, i_{1t} shows total investment expenditures for the first type of capital. From (2.9), (2.10), and (2.11) comes an alternative expression for i_{1t}:

$$i_{1t} = g_0 k_{1t+J} + g_1 k_{1t+J-1} + \cdots + g_J k_{1t} = g(L)k_{1t+J} \tag{2.12}$$

where $g_0 = \phi_{J-1} - (1-\delta_1)\phi_J$, $g_1 = \phi_{J-2} - (1-\delta_1)\phi_{J-1}$, ..., and $g_J = -(1-\delta_1)\phi_1$. This specification of the investment technology differs from Kydland and

Prescott's only in terms of separately modeling the behavior of structures and the behavior of equipment. Kydland and Prescott considered a single type of productive capital subject to the time-to-build assumption.

2.2 The Approximate Equilibrium

The equivalence of competitive equilibrium and Pareto optimum for this economy produces equilibrium allocations. This involves maximizing the expected discounted utility of the representative consumer subject to the constraints imposed by the technology and the information structure:

$$\text{(2.13)} \qquad \underset{\{n_t\},\{s_{Jt}\},\{i_{2t}\}}{\text{maximize}} \; E\left(\sum_{t=0}^{\infty} \beta^t u[f(\lambda_t, n_t, k_{1t}, k_{2t}) - i_{1t} - i_{2t},\; l_t^*] \,|\, \Omega_0\right)$$

subject to

$$i_{1t} = \sum_{j=1}^{J} s_{jt}$$

$$k_{1t+1} = (1-\delta_1)k_{1t} + s_{1t}$$

$$k_{2t+1} = (1-\delta_2)k_{2t} + i_{2t}$$

$$s_{j,t+1} = s_{j+1,t} \qquad \text{for } j = 1, \dots, J-1$$

$$l_t^* = \begin{cases} 1 - (1-\eta)n_t - \eta(1-\eta)a_t \text{ with } a_{t+1} = \eta a_t + n_t \\ \text{or} \\ l_t + \beta_1 l_{t-1} \end{cases}$$

$$n_t \geq 0, \; s_{Jt} \geq 0, \; i_{2t} \geq 0$$

given the initial conditions s_{j0} for $j = 1, \ldots, J - 1$; k_{10}; k_{20}; λ_0; and either a_0 or (l_{-1}) and given the stochastic law of motion for λ_t. In (2.13), the constraint $c_t = Q_t - i_{1t} - i_{2t}$ has been used to eliminate c_t from the representative consumer's utility function, while $E(\cdot|\Omega_0)$ denotes expectation conditional on information available at time zero. When solving this problem, Kydland and Prescott assumed that the labor supply/labor input choice is based on a smaller information set than the investment decisions. They made the former conditional on observations of past values of the technology shock and a noisy indicator of its current value and the latter on observations of current as well as past output and hence the technology shock. I retain the two-stage information structure but eliminate the indicator shock because its variance is not identified. Here the current labor/leisure allocation is based only on observations of the technology shock up to period $t - 1$.

For the functional forms assumed here, (2.13) does not admit a closed-form solution. To get a solution that depends on the parameters of preferences and technology and that expresses the variables of the model as linear functions of the innovation to the technology shock, I approximate the social planner's problem in (2.13) as Kydland and Prescott suggested. Their procedure replaces the single-period utility function in (2.13) by one quadratic in k_{1t}, k_{2t}, $i_{1t} + i_{2t}$, n_t, and either a_t or n_{t-1} and assumes that the long-run equilibrium of the original economy is characterized by certainty equivalence.[3/]

In Appendix A, I use the methods of Hansen and Sargent [1981] to derive the approximate equilibrium laws of motion for the model's variables. Equation (2.14) gives the implied representation for the vector of stock variables $x_t = (k_{1t+J}, a_{t+1}, k_{2t+1})'$, while (2.15) is the corresponding representation for the vector of flow variables $y_t = (i_{1t}, n_t, i_{2t}, Q_t)$:

$$(2.14)\qquad x_t = x_{t-1} + C(L)^{-1}B(L)\varepsilon_t$$

$$(2.15)\qquad y_t = y_{t-1} + [E(L)C(L)^{-1}B(L) + G]\varepsilon_t = y_{t-1} + H(L)\varepsilon_t.$$

In these equations, $C(L)$ is a J^{th}-order matrix polynomial in nonnegative powers of L, such that the roots of det $C(z) = 0$ are greater than $\beta^{-1/2}$ in modulus. $B(L)$ and $E(L)$ are, respectively, first- and J^{th}-order matrix polynomials in nonnegative powers of L and have elements of order 3 x 1 and 4 x 3, while G is a 4 x 1 vector of constants. Finally, $H(L)$ is defined from the first matrix polynomial of equation (2.15).

3. The Econometric Model

The representations for x_t and y_t are useful only if they let us take the model to data. As they stand, however, neither (2.14) nor (2.15) do that because both $\{x_t - x_{t-1}\}_{t=0}^{\infty}$ and $\{y_t - y_{t-1}\}_{t=0}^{\infty}$ are stochastic processes with perfectly correlated elements. Equivalently, their variance-covariance matrix is singular. This occurs because not enough exogenous shocks drive the endogenous series. Since the singularity of these processes is sure to be rejected by actual data, the model must have shocks added in order to have empirical content.

3.1 Specification

Consider equation (2.15), and let $v_t = (v_{1t}, \ldots, v_{4t})'$ denote the vector of measurement errors for the differences ξ_t of the measured series. The econometric specification derived from equation (3.1) assumes that those differences are noisy measures of the differences of the true series, $y_t - y_{t-1}$, that is,

$$(3.1)\qquad \xi_t = y_t - y_{t-1} + v_t = H(L)\varepsilon_t + v_t = [E(L)C(L)^{-1}B(L) + G]\varepsilon_t + v_t.$$

In this specification, the individual measurement errors v_{it} for $i = 1, \ldots, 4$ are defined to have mean zero and variance σ_i^2 and to be serially uncorrelated. Also, $E(\varepsilon_t, v_{it}) = 0$ and $E(v_{it}, v_{jt}) = 0$ for $i \neq j$ and $i, j = 1, \ldots, 4$. *Under the latter assumption,* (3.1) *provides* a one-factor or single-index representation for the differences ξ_t of the observable time series.4/ Here any correlation between the elements of ξ_t arises from their mutual dependence on the unobservable innovation to productivity while the measurement errors constitute specific disturbances.

I will also estimate a simple alternative to (3.1) which imposes the single-factor structure but does not place any cross-equation restrictions on the matrix lag polynomial linking the common shock to the observable series. This model is described by

$$(3.2) \qquad \xi_t = \overline{H}(L)\bar{\varepsilon}_t + \bar{v}_t$$

where $\overline{H}(L) = (\bar{h}_1(L), \ldots, \bar{h}_4(L))'$; $\bar{h}_i(L) = b_i/(1+\bar{a}_{i1}L+\bar{a}_{i2}L^2)$; $\bar{v}_t = (\bar{v}_{1t}, \ldots, \bar{v}_{4t})'$; $E(\bar{v}_{it}, \bar{v}_{it-k}) = 0$ for $i = 1, \ldots, 4$ and $k > 0$; $E(\bar{v}_{it}^2) = \bar{\sigma}_i^2$ for $i = 1, \ldots, 4$; and $E(\bar{\varepsilon}_t^2)$ is normalized as one. The econometric specifications described by equations (3.1) and (3.2) are not nested--the latter imposes no restrictions across the rows of $\overline{H}(L)$, while the former allows for longer lag lengths in the polynomials describing the moving average and autoregressive parts of ξ_t relative to the second-order ARMA representation implicit in (3.2)--and so cannot be easily compared using the likelihood ratio criterion. However, estimates of (3.2) can be used as a reference point to illustrate the dynamic properties of (3.1).

3.2 The Estimation Strategy

A frequency domain approximation to the exact likelihood function for a sample of observations on the $\{\xi_t\}_{t=0}^{\infty}$ process is used to estimate the unknown parameters in (3.1) and (3.2). Define the vector of parameters in (3.1) by $\underline{\theta}$, which includes the structural parameters $(\beta, \gamma, \eta, b, \theta, \nu, \sigma, \phi_1, \ldots, \phi_J, \delta_1, \delta_2, \bar{\lambda})'$ and the variances of the common shock and of the measurement error shocks $(\sigma_\varepsilon^2, \sigma_1^2, \ldots, \sigma_4^2)'$. Similarly, define the unknown parameters in (3.2) by $\underline{\bar{\theta}}$. Appendix B derives the approximate likelihood function and describes how it is calculated for each set of values for $\underline{\theta}$ and $\underline{\bar{\theta}}$.

4. The Data

The variables I use in estimation consist of output, total hours worked, and investment expenditures for structures and equipment. The period of observation is 1947:I-1981:IV, and all series are measured at quarterly rates. The output and investment series are from the National Income and Product Accounts. Investment in structures is measured as the sum of investment in residential and nonresidential structures, including farm and nonfarm structures. Investment in equipment is similarly defined as the sum of investment expenditures for residential and nonresidential producers' durable equipment. The investment and output series are seasonally adjusted and expressed in 1972 dollars, with corresponding CITIBASE codes GIS72, GIRF72, GIRU72, GIPD72, GIRD72, and GNP72. The quarterly series for hours worked is derived from the three-month averages of the seasonally adjusted CITIBASE series "man-hours employed per week for all workers in all industries" (LHOURS). The number of weeks in a quarter is taken to be 12.75. In the data, total hours vary due to the number of persons working as well as the hours worked by those already employed. With identical consumers, the former type of variation is not in the theoretical hours series. While inconsistent with the model's assumptions, I use this measure of total hours because it represents more accurately the fluctuation in aggregate hours worked. According to the representative agent framework, the theoretical series correspond to per capita values of the measured series. Hence, all series are divided by the Bureau of Census measure of the resident adult population (CITIBASE Code POPT16) to obtain per capita values. First-differencing the per capita series provides the series for estimation.

5. Empirical Results

5.1 Exploratory Analysis

Before taking (3.1) and (3.2) to data, I estimate a model which allows a simple test of the hypothesis that a single unobservable index accounts for the covariation among the elements of ξ_t.[5/] Such a model is defined by

$$(5.1) \qquad S_{\xi}(\omega) = \tilde{H}(\omega)\tilde{H}(\omega)^* + \tilde{V}(\omega).$$

This model restricts the $\{\xi_t\}_{t=0}^{\infty}$ process only by attributing the variation in ξ_t at frequency ω to the covariation among the latent common factors of a low dimensional vector and to the variation in a vector of specific disturbances, which are mutually uncorrelated and uncorrelated with the common factor. In (5.1), the variance of ξ_t at frequency ω due to the common and specific factors is defined by $\tilde{H}(\omega)\tilde{H}(\omega)^*$ and $\tilde{V}(\omega)$, respectively.

Table 1 reports the chi-square statistics for the test of the hypothesis that $S_{\xi}(\omega)$ decomposes according to a one-noise model versus the alternative that $S_{\xi}(\omega)$ is merely positive-definite. The degrees of freedom for the chi-square statistics at the different frequencies is 5 (following Geweke [1977]). The overall likelihood ratio test statistic is distributed asymptotically as a chi-square random variable with 30 degrees of freedom. The corresponding marginal significance levels are in this table too. According to these statistics, the null hypothesis that a single factor accounts for the covariation among the elements of the differenced series ξ_t cannot be rejected at conventional significance levels. This holds when the restrictions of (5.1) are considered overall as well as at the individual frequencies. Table 1 also presents the coherences of the elements of ξ_t with the common unobservable factor at the six listed frequencies. From (5.1), the coherence for the i^{th} element of ξ_t at frequency ω is defined as

$$(5.2) \qquad \mathrm{coh}_i(\omega) = \frac{\tilde{H}_i(\omega)\tilde{H}_i(\omega)^*}{[S_\xi(\omega)]_{ii}}$$

where $\tilde{H}_i(\omega)$ and $[S_\xi(\omega)]_{ii}$ denote the corresponding i^{th} elements of the 4 x 1 and 4 x 4 matrices $\tilde{H}(\omega)$ and $S_\xi(\omega)$. The single factor has the most explanatory power at the low frequencies, accounting for more than 50 percent of the variation in the cyclical components for all series with periods of 14 quarters or more and, among the different series, is most successful in explaining the variation for the differenced values of per capita output.

The model described by equation (5.1) may also be used to test for the existence of serial correlation in the elements of $\tilde{v}_t$. Table 2a presents estimates of $\tilde{V}_i(\omega)$ which show the variance in each series attributed to its own specific disturbance. Under the assumption that $\{\tilde{v}_t\}_{t=0}^{\infty}$ for $\tilde{v}_t = (\tilde{v}_{1t},\ldots,\tilde{v}_{4t})'$ is a serially independent process, $\tilde{V}(\omega) = (\tilde{V}_1(\omega),\ldots,\tilde{V}_4(\omega))'$ should be constant across all frequency bands. Consequently, testing the null hypothesis that $\{\tilde{v}_t\}_{t=0}^{\infty}$ is serially uncorrelated against an alternative of arbitrary serial correlation is equivalent to testing the five equality constraints, $\tilde{V}_i(\omega_1) = \tilde{V}_i(\omega_2)$, $\tilde{V}_i(\omega_2) = \tilde{V}_i(\omega_3)$, $\tilde{V}_i(\omega_3) = \tilde{V}_i(\omega_4)$, $\tilde{V}_i(\omega_4) = \tilde{V}_i(\omega_5)$, and $\tilde{V}_i(\omega_5) = \tilde{V}_i(\omega_6)$ for $i = 1, \ldots, 4$. Here $\omega_1, \ldots, \omega_6$ are the frequencies at which (5.1) is estimated. The relevant test statistics are reported in Table 2b.[6] Given the small values of t_i for i = 1, 2, 4, the null hypothesis cannot be rejected at conventional significance levels for the differences of the hours, output, or investment in structures series. Still, the value of t_3 equal to 12.57 implies that the variation in $\tilde{V}_3(\omega)$ across frequencies is consistent with serial independence in the measurement error for the differences of equipment only at the 3 percent level. Yet the reported estimates of $\tilde{V}_3(\omega)$ do not suggest a convenient way to parametrize the serial correlation in $\tilde{v}_{3t}$. Hence, in what follows, I assume that $\tilde{v}_{3t}$ is serially uncorrelated as well.

The above results suggest that imposing the single-factor structure in (3.1) and assuming serial independence for the vector of measurement errors v_t should not lead to the rejection of (3.1) for the purpose of describing the joint behavior of the investment, hours, and output series. To determine whether the remaining restrictions of the economic model are supported by the data, I now turn to the estimation of (3.1).

5.2 Estimates of Equation (3.1)

To estimate (3.1), the likelihood function (B.2) in Appendix B is maximized with respect to the parameter vectors $\underline{\theta}$. I do not estimate all elements of $\underline{\theta}$. Recall that J, the number of periods required to build productive capital, is not determined by the model and must be specified before estimation. As did Kydland and Prescott [1982], I set J equal to 4. Some justification for doing so exists in a survey by Mayer [1960], who found that the average time required for the completion of projects involving plants and structures was three to four quarters. This duration would seem a reasonable assumption for the completion of residential structures as well. The subjective discount factor is set at 0.9909, implying a constant real interest rate of 4 percent per year. The depreciation rates for structures and equipment are set at $\delta_1 = 0.02$ and $\delta_2 = 0.03$. These values are close to some estimates in the literature, albeit obtained from different models. For example, Jorgenson and Hall [1967] reported annual depreciation rates of 0.1471 and 0.1923 for equipment in manufacturing and nonfarm nonmanufacturing industries. The analogous rates for structures are 0.0625 and 0.0694.[7/]

Table 3 reports the results of estimating (3.1). The estimates in column (a) are derived by allowing the risk aversion parameter γ to be free and assuming an infinite distributed lag characterizing the service flow from current and past leisure choices. Column (b) constrains γ to equal unity and assumes that $l_t^* = l_t + \beta_1 l_{t-1}$.

Several interesting findings emerge from Table 3. One concerns the parameter γ, which determines the curvature of the one-period utility function. In column (a), it is estimated as 1.6589. With a standard error of 0.658, this estimate may be interpreted as evidence for the true value of γ being equal to unity, the latter value implying risk neutrality in terms of the composite consumption good $c_t^b l_t^{*1-b}$. For the class of utility functions I consider, the coefficient of relative risk aversion, defined by the expression $1 - \gamma$, is also equal to the inverse of the intertemporal substitution elasticity for the composite consumption good. Hence, this estimate of γ may also be interpreted as evidence that this elasticity is large--infinite, in fact. Furthermore, an estimate of 1.6589 for γ implies an estimate of 1.14 for $\gamma(1-b)$, the coefficient for leisure services l_t^* in the single-period utility function. Although its standard error is large at 0.66, this estimate for $\gamma(1-b)$ could point to a possible violation of the convexity of preferences with respect to l_t^*.

Another conclusion clear from Table 3 is that estimation of the model under two different specifications for the durability of leisure yields fairly similar estimates for many of the underlying parameters. Of these, the estimates of 0.9884 and 0.9812 for the parameter θ, which determines the share of labor in the aggregate production function, deserve mention. Since the value of θ is restricted to lie between zero and one during the estimation, these estimates, which are close to θ's upper bound, suggest that the aggregate production function will display increasing returns to scale with respect to labor in the absence of such a constraint on θ. Another similarity between the results of columns (a) and (b) is the estimates of the share parameter b in the one-period utility function: they are 0.3142 and 0.2530, respectively. Also, the variance σ_ε^2 of the innovation to the technology shock is around 0.001 in both cases, implying that the variability of $\lambda_t - \lambda_{t-1}$ around a mean of zero is small and equal to roughly 0.4 percent per year.[8/] Similar evidence also emerges for the time-to-build feature for investment in structures: the estimates of ϕ_j for $j = 1, \ldots, 4$ increase with j, implying that a declining proportion of resources are allocated to the investment projects closer to completion. Judging by their standard errors, these estimates are also consistent with the true values of the time-to-build coefficients being different from 0.25. Of the remaining technology parameters, the estimates of the parameter σ determining the share of equipment in the composite capital good are both small but imprecise, given their standard errors. Finally, the estimates of ν imply that the quantity $1/(1+\nu)$, which measures the elasticity of substitution between the two types of capital in the aggregate production function, is estimated to be less than one. The estimates of σ and ν must be viewed cautiously, however: with θ driven close to a value of one, these parameters become only weakly identified because capital goods matter little in the aggregate production function.

The results of columns (a) and (b) of Table 3 differ in one important way. This concerns the estimates of the parameters η and β_1, which determine the effect of past leisure choices on current utility according to the specifications $l_t^* = 1 - (1-\eta)n_t - \eta(1-\eta)a_t$ and $l_t^* = l_t + \beta_1 l_{t-1}$, respectively. In column (a), η is estimated as 0.00085 and has a standard error consistent with the true value of η being equal to zero. But in column (b), the estimate of β_1 is 0.8064, large relative to its standard error of 0.130. Under the first specification of preferences, leisure does not appear as a durable good; under the second, leisure consumed with a one-period lag seems to have a positive effect on current utility.

To better account for the differences between the two sets of results and to get sharper conclusions about the effects of some of the key parameters γ, η, β_1, and θ, I estimate two additional specifications. The first sets γ equal to one and assumes that preferences are time separable with respect to leisure, that is, $l_t^* = l_t$. In addition to these restrictions, the second specification sets θ equal to 0.64. The value of the maximized likelihood function under the first specification is -1,844.386, while under the additional restriction of $\theta = 0.64$ it is -1,894.521. The estimates of the remaining parameters (not reported here) are, on the whole, similar to those from the specifications of Table 3.

According to these results, the values of the chi-square statistics for the tests of the single restriction of β_1 equal to zero and of the joint restrictions of γ equal to one and η equal to zero are 24.287 and 23.6, respectively. Since the estimate of η is close to zero and has a relatively large standard error, the rejection of $\gamma = 1$ and $\eta = 0$ seems to arise from the rejection of $\gamma = 1$. Hence, the result that β_1 is significantly different from zero is somewhat surprising. One explanation is that lagged values of leisure do have an effect on current utility, but that the parametrization for l_t^* given by $l_t^* = 1 - (1-\eta)n_t - \eta(1-\eta)a_t$ forces the estimate of η to zero in order to make the coefficient on current leisure (or current labor supply) equal to one. When l_t^* is defined as $l_t^* = l_t + \beta_1 l_{t-1}$, the coefficient on current leisure is a priori restricted to one. Hence, the estimate of β_1 picks up only the effect of past leisure choices and specifies this to be positive. An alternative interpretation of these results is that identifying certain classes of models using only quarterly aggregate time series is difficult. In this respect, it seems suggestive that the likelihood value L_T^* when $\gamma = 1.6589$ and $\eta = 0.00085$ is $-1,832.2423$, while it is $-1,832.5860$ when $\gamma = 1$ and $\beta_1 = 0.8064$. While the likelihood values under the non-nested specifications for l_t^* cannot be directly compared, they nevertheless suggest that the covariation in the hours, output, and investment series may be rationalized by two alternative specifications of preferences. According to one, preferences are time separable with respect to leisure, but there is some type of increasing returns or nonconvexity with respect to per capita leisure hours. According to the other, preferences are convex with respect to consumption and leisure, but past leisure choices affect current utility.

The results of the test for the hypothesis that $\theta = 0.64$ show no such ambiguity. A comparison of the values of the maximized likelihood functions under the restriction that $\gamma = 1$ and $\eta = 0$ versus the restriction that, besides that, $\theta = 0.64$ shows that the value of a chi-square statistic with one degree of freedom is equal to 100.269. Consequently, despite the rather large standard error for the θ estimate reported in column (a) of Table 3, which seems to suggest that an estimate of 0.9884 is consistent with the true value of θ being equal to 0.64 as well as with a value of one, the data strongly reject the hypothesis that $\theta = 0.64$. Furthermore, this restriction is rejected not only when tested separately, but also when tested as part of the joint hypothesis that $\gamma = 1$, $\eta = 0$ (or $\beta_1 = 0$), and $\theta = 0.64$ against the alternatives presented in Table 3. The values of the relevant chi-square statistics, having three and two degrees of freedom, respectively, are 124.5564 and 123.8490.

In summary, three main conclusions emerge from formally estimating the model. These are the findings of an estimate close to unity for the elasticity of output with respect to labor, estimates of the parameters γ and b consistent with large intertemporal substitution elasticities with respect to a composite consumption good and with possible nonconvexity of a representative consumer's preferences with respect to per capita leisure hours, and finally, some ambiguity about the importance of the durability of leisure.

At this point, several questions arise. How well does the model restricted according to equation (3.1) explain the joint movement of the hours, output, and investment series relative to other unrestricted specifications? Which, if any, of Kydland and Prescott's conclusions about the performance of their model are supported by these results?

5.3 Time Series Properties of Models (3.1) and (3.2)

For the model restricted according to equation (3.1), I compute diagnostics under the set of estimates obtained by setting $\gamma = 1$ and $l_t^* = l_t$. These estimates, together with the implied equilibrium laws of motion for $y_t - y_{t-1}$, are reported in Appendix C. Table 4 presents the estimates used to calculate such diagnostics for the simple time series representation for ξ_t described by equation (3.2).

Table 5 shows the responses of the output, hours, and investment series to a unit impulse in the innovation to the technology shock ε_t and the comparable responses derived from representation (3.2). The impulse responses based on (3.1) show that the initial response of all variables to a unit change in productivity is positive. However, the subsequent responses differ for the output and hours series, on the one hand, and the investment series, on the other. An increase in ε_t initially increases the marginal product of both types of capital and consequently the values of the optimal capital stocks. With lags of four periods in the production of structures and interrelated decisions for the optimal capital stocks, both investment series reach a peak after five quarters and then drop. Despite small oscillations, though, both the output and hours series climb to permanently higher values after the initial shock. The permanent increase in hours arises because larger values of the capital stocks increase the marginal product of labor. Consequently, although (3.2) implies simple behavior, the underlying economic model, whose restrictions are described by (3.1), seems able to generate interesting dynamic interrelationships among the series.

Despite this potential for producing complicated dynamics, however, the coherences of Table 6 show that the underlying economic model can account for only a small proportion of the variance of the two investment series. Although the common factor in (3.2) accounts for at least 50 percent of the variance for the components of all series with periods of seven quarters or more, the explained variance for both investment series drops drastically, both overall and at the low frequencies, when the restrictions described by equation (3.1) are imposed. For example, Table 6a shows the underlying economic model can explain at most 10 percent of the variation in the components of the equipment series with periods of 14 quarters or more.

The small coherences of Table 6a do not necessarily provide evidence against the representation described by (3.1), however. That model may be correct, but the differences of the investment series contain large measurement errors, especially at the low frequencies. To account for this possibility, recall the estimates of the exploratory dynamic factor model in Tables 1 and 2. There the specific disturbances of the different series are generally small and serially uncorrelated. If the behavior of the specific disturbances in (5.1) indicates anything about the behavior of the measurement errors of (3.1), the variance attributed to the latter should also be small and constant across frequencies. The econometric model described by (3.1) already assumes that the measurement errors v_{it} for $i = 1, \ldots, 4$ are serially uncorrelated and have constant variances σ_i^2 for $i = 1, \ldots, 4$ across all frequencies. So the only remaining implication of comparing the results of Tables 2 and 6a is that the variance attributed to the specific disturbances according to the estimates of (3.1) should be small, as they are according to the estimates of (5.1). Another way of making this point is to notice that a consistent estimate of the population variance, at each frequency, for the elements of the $\{\xi_t\}_{t=0}^{\infty}$ process may be obtained in at least two ways: from estimates of the total variance, of each frequency, computed under the restrictions of equations (3.1), (3.2), or (5.1) or from an unrestricted spectral estimator computed under the assumption that $S_\xi(\omega)$ is positive-definite across all frequencies. Hence, if the restrictions of the models described by equations (3.1) or (3.2) for the variances of each series are at all supported by the data, then the estimates of $H(\omega)H(\omega)^*$ and $\overline{H}(\omega)\overline{H}(\omega)^*$--which show the variance attributed to the common latent factors in the spectral estimates restricted according to equations (3.1) and (3.2)--should be close to, and have the same shape as, an unrestricted estimate computed under the assumption that $S_\xi(\omega)$ is positive-definite across frequencies.

Figures 1-4 show that, for the model represented by equation (3.1), this is not true. There the plots of the logarithms of the diagonal elements of $H(\omega)H(\omega)^*$ and $\overline{H}(\omega)\overline{H}(\omega)^*$ are denoted R and F and the plots of the logs of the unrestricted spectral estimates and their 95 percent confidence intervals are denoted U and *.[9/] For large bands of frequencies, the restricted spectral estimates of the investment series neither have the same shape as nor fall within the 95 percent confidence intervals of the unrestricted estimates. Consequently, the small coherences for the investment series implied by model (3.1) seem not to be due to the nature of the measurement error in these series. Instead, these coherences seem to be small because the underlying economic model cannot explain the variability of the investment series. By contrast, not only are the equation (3.1)-restricted coherences of the hours and output series large, with close to 40 and 90 percent of the variation in their components with periods of seven quarters or more attributed to variation in the common shock, but their restricted spectral estimates, computed from the appropriate diagonal elements of $H(\omega)H(\omega)^*$, fall within the confidence intervals of the unrestricted estimates over many frequencies. In particular, the output series generated by the underlying economic model mimics very closely the behavior of the differences of the actual output series.

5.4 Comparison with Kydland and Prescott

The results reported so far show that the underlying economic model fails to capture the variability of the differences of the two investment series. It does, however, seem able to generate a large proportion of the variability of the differences of per capita hours under a time-separable specification of preferences. These results are contrary to some of the main conclusions of Kydland and Prescott. But can their conclusions be directly compared with mine? They adopted a very simple metric to determine the values of the unknown parameters of the model: they fixed some according to a priori information and chose the rest according to a rough grid search designed to match the variances of a set of detrended series, the contemporaneous correlation of the same series with a detrended measure of output, and the five autocorrelations of detrended output generated from the model with the values of the same statistics computed from actual data. Moreover, they detrended their series with a complicated two-sided filter[10/] and computed the statistics implied by their model from a singular representation, such as those described by equations (2.14) or (2.15). By contrast, I augmented the representations in (2.14) and (2.15) with measurement errors to derive a well-defined econometric model and chose values for a major subset of the unknown parameters by matching all the covariances for the differences of the four key series according to the metric defined by maximum likelihood estimation.

In terms of determining whether the restrictions of the underlying economic model were at all supported by the data, however, I used criteria quite similar to Kydland and Prescott's. More precisely, I asked, first, if a significant proportion of the total variance for the differences of each series could be explained by variation attributed to the single common factor which, according to equation (3.1), is defined as the innovation to the technology shock and, second, if the underlying economic model could reproduce some of the time series properties of the different series. Because Kydland and Prescott used a singular model to compute the theoretical variance for each series, their measure of this variance corresponds, in my framework, to the variation in each series attributed to the single common factor. For the model described by equation (3.1), it is proportional to the sum of the diagonal elements of $H(\omega)H(\omega)^*$ across frequencies. Consequently, examining the coherences of Table 6a or comparing the values of the diagonal elements of $H(\omega)H(\omega)^*$ across frequencies with the values of some unrestricted spectral estimator at those frequencies is similar to Kydland and Prescott comparing the theoretical variance of the different series with the sample variance of these series. Similarly, examining the serial correlation properties of the different series by looking at autocorrelations is similar to examining the low frequency variation in each series which can be attributed to the innovation to the technology shock.

The final unanswered question is, why did Kydland and Prescott reach their conclusions? One important reason is that they assumed a priori the existence of a well-behaved neoclassical production technology describing the relationship between aggregate output and the inputs of labor and two types of capital, the latter consisting of the stock of a single aggregate capital good and the stock of aggregate inventories. But what my results seem to show is that, even if inventories are not considered an input and the aggregate capital stock is disaggregated as the stock of structures and equipment, the fact that, in the absence of arbitrary adjustments for capacity utilization, deviations of per capita output around some trend fluctuate quite a bit while deviations of per capita hours around its trend do not leads to an estimate of unity for the elasticity of output with respect to labor, just as in other studies of aggregate production relationships (for example, Lucas [1970] and Tatom [1980]). This finding is what causes the model to fail to explain the behavior of the investment series.

That can be demonstrated more formally by considering the expressions for the spectral estimates of the investment series restricted according to equation (3.1). These are derived from the laws of motion for the differences of the investment series, $i_{1t} - i_{1t-1} = g(L)(k_{1t+J}-k_{1t+J-1})$ and $i_{2t} - i_{2t-1} = [1-(1-\delta_2)L](k_{2t+1}-k_{2t})$, as

$$(5.3) \qquad S_{11}(\omega_j;\underline{\theta}) = |g(\omega_j)|^2 S_{k_1}(\omega_j;\underline{\theta}) + \sigma_1^2$$

$$(5.4) \qquad S_{33}(\omega_j;\underline{\theta}) = |1-(1-\delta_2)e^{i\omega_j}|^2 S_{k_2}(\omega_j;\underline{\theta}) + \sigma_3^2.$$

Here $S_{11}(\omega_j;\underline{\theta})$ and $S_{33}(\omega_j;\underline{\theta})$ are the diagonal elements of $S(\omega_j;\underline{\theta})$ corresponding to the two types of investment, $S_{k_1}(\omega_j;\underline{\theta})$ and $S_{k_2}(\omega_j;\underline{\theta})$ are the spectral densities for the capital stock series, and $|g(\omega_j)|^2$ and $|1-(1-\delta_2)e^{i\omega_j}|^2$ represent the squared modulus of the Fourier transforms of $g(L)$ and $[1-(1-\delta_2)L]$, respectively. These equations show that the proportion of variance explained by the underlying economic model at each frequency for the investment series is proportional to that for the capital series. The terms $|g(\omega_j)|^2$ and $|1-(1-\delta_2)e^{i\omega_j}|^2$ may have an effect in reducing the low frequency power of the investment series. (For example, $|g(\omega_j)|^2$, which describes the effect of the time-to-build assumption, can be shown to be less than one across all frequencies and reaching a minimum at frequency zero.) The effect of such terms should be negligible, however, for large values of $S_{k_1}(\omega_j;\underline{\theta})$ and $S_{k_2}(\omega_j;\underline{\theta})$. Still, with an estimate of θ close to one, the model does not assign a productive role to capital stocks in the aggregate production function. Thus, it does not generate large values for $S_{k_1}(\omega_j;\underline{\theta})$ and $S_{k_2}(\omega_j;\underline{\theta})$ and ends up attributing much of the variation in the investment series to variation in the measurement errors.

Kydland and Prescott did not let their data freely determine the parameters of the aggregate production function. Instead, they restricted the parameter governing the elasticity of output with respect to labor to equal 0.64. Then they chose the value of γ, the parameter which determines the coefficient of relative risk aversion in the composite good $c_t^b l_t^{*1-b}$, to equal -0.5 (a value of zero implying a logarithmic form of the single-period utility function in this good). They then concluded that the model explains investment, but that hours fluctuate less than productivity when preferences are assumed to be time separable with respect to leisure. By contrast, my Table 3 shows, at best, mixed results for a non-time-separable specification of preferences with respect to per capita leisure hours. Nevertheless, I do show that a large fraction of the low frequency variation in the differences for per capita hours can be generated under the assumption that past leisure choices have no effect on current utility.

6. Conclusion

One conclusion that can be drawn from this paper concerns the ability of the underlying economic model to explain, even in relatively rough terms, the joint movement of some key aggregate series. The paper shows that this model leaves unexplained a large portion of the variance of two investment series, namely, investment in structures and investment in equipment. It argues that the model fails this way because, at a more fundamental level, the model cannot rationalize the joint behavior of aggregate output and hours under the assumption of a well-behaved neoclassical production technology. A more positive conclusion can also be drawn from the paper, however. This is that an interesting class of dynamic equilibrium models--models which can be used to study cyclical fluctuations, growth, and alternative theories of investment and of aggregate consumption and labor supply--can be estimated and tested using classical methods.

Footnotes

*I thank Lars Peter Hansen, Finn Kydland, Robert Litterman, Robert Miller, Christopher Sims, and Kenneth Singleton for their suggestions. In addition, I am grateful to Kathy Rolfe for excellent editorial assistance. An earlier version of this paper was presented at seminars at the Massachusetts Institute of Technology; Northwestern University; and the Universities of California (Los Angeles), Chicago, Minnesota, Rochester, and Southern California; as well as at the 1984 National Bureau of Economic Research Conference on Business Fluctuations and the 1984 North American Summer Meeting of the Econometric Society. The views expressed herein are my own and not necessarily those of any of the above, the Federal Reserve Bank of Minneapolis, or the Federal Reserve System.

1/ Long and Plosser [1983] present another recent example of a real business cycle model.

2/ Some empirical support for specifying the technology process this way comes from an unexpected quarter. Analyzing the residual from Solow's [1957] aggregate production function, Nelson and Plosser [1982] found that it could be well described as a random walk. But such an analogy is somewhat tenuous because, as will become clear shortly, the technology shock in this paper is unobservable by an econometrician, so cannot be measured as the residual from an estimated production function.

3/ Appendix A describes the approximation procedure in more detail.

4/ See Sargent and Sims [1977] for a further description of such models.

5/ Geweke [1977] provides a comprehensive discussion of identification, estimation, and tests of models similar to mine. Briefly, to implement the test I describe in section 5.1, calculate the finite Fourier transform of the sample of observations $\{\xi_t\}_{t=1}^{T}$ at the harmonic frequencies $\omega_j = 2\pi j/T$ for

$j = 0, \ldots, T-1$. Denote this Fourier transform $w(j,T)$. Let ω_q be the frequency at which my equation (5.1) is estimated and ℓ_q any positive integer. Also define the set of ℓ_q vectors $w(j_T+1,T), \ldots, w(j_T+\ell_q,T)$, where j_T is chosen to minimize $\sum_{i=1}^{\ell_q} |2\pi(j_T+i)-\omega_q T|$. Then it can be shown that the joint distribution of the ℓ_q vectors converges to that of ℓ_q independent vectors, each possessing a complex normal distribution with mean zero and covariance matrix $S_\xi(\omega_q)$. If $\omega_q = 0$ or $\omega_q = \pi$, then this distribution is ordinary-normal. The complex likelihood function for $w(j_T+i,T)$ for $i = 1, \ldots, q$ is defined, for example, by Geweke [1977]. Here, $q = 1, \ldots, Q$ defines the frequency band consisting of ℓ_q ordinates across which this complex likelihood function is evaluated. I estimate (5.1) at six frequencies: 0.0694π, 0.2361π, 0.4028π, 0.5694π, 0.7361π, 0.9028π. The finite Fourier transform of the sample $\{\xi_t\}_{t=1}^{T}$ is calculated at a total of 72 harmonic frequencies between 0 and π. Out of this total of 72 ordinates, estimates of $\tilde{H}(\omega)$ and $\tilde{V}(\omega)$ are obtained from disjoint bands of 9 periodogram ordinates each. In practice, all estimates and tests of the unrestricted frequency domain factor model are obtained using Robert Litterman's INDEX program.

6/ Let $D(\omega_q) = (2/\ell_q)E(\omega_q)$, where $E(\omega_q)$ is the inverse of the matrix of second derivatives of the complex likelihood function for $w(j_T+1,T)$ for $i = 1, \ldots, \ell_q$ with respect to the unknown parameters of $\tilde{H}(\omega_q)$ and $\tilde{V}(\omega_q)$ in the q^{th} frequency band. Let R be the $(Q-1) \times Q$ matrix with $[r_{qq}] = 1$, $[r_{q,q+1}] = -1$, and zero elsewhere. Also, define the $Q \times 1$ vector $\tilde{V}_j$ as $\tilde{V}_j = [\tilde{V}_j(\omega_1), \tilde{V}_j(\omega_2), \ldots, \tilde{V}_j(\omega_Q)]'$. Then a statement of the null hypothesis in the text corresponds to the statement that $R\tilde{V}_j$ should be equal to the $(Q-1) \times 1$ vector of zeros. This linear constraint can be tested using the test statistic $t_j = (R\tilde{v}_j)'(RGR')^{-1}R\tilde{v}_j$, where G is the $Q \times Q$ diagonal matrix containing the estimated variances of $\tilde{V}(\omega_q)$ for $q = 1, \ldots, Q$ obtained from $D(\omega_q)$ on its diagonal. The required test of no serial correlation in $\{\tilde{v}_{jt}\}_{t=0}^{\infty}$ is then obtained by noting that under the null hypothesis t_j is distributed as $\chi^2(Q-1)$. (See Geweke and Singleton [1981] for a further description of this test.)

[7]The approximate likelihood function implied by models (3.1) and (3.2) is maximized using the DFP (Davidon-Fletcher-Powell) option of the Goldfeld-Quandt nonlinear optimization routine. DFP uses a hill-climbing algorithm based on the true gradient of the likelihood function and an approximation to the inverse of the matrix of second derivatives. Since the form of the likelihood (B.2) under (B.4) and (B.5) makes obtaining even analytic first derivatives difficult, the gradient is also evaluated numerically by DFP.

The DFP algorithm is allowed to conduct the numerical search over an unrestricted parameter space. However, to ensure the existence of the deterministic equilibrium and also the existence of a solution to the dynamic optimization problem generating the equilibrium laws of motion, the values of the parameters which actually enter the likelihood function under model (3.1) are constrained as follows: $\eta = [1+\exp(H)]^{-1}$; $b = [1+\exp(B)]^{-1}$; $\theta = [1+\exp(T)]^{-1}$; $\sigma = [1+\exp(S)]^{-1}$; $\phi_i = \exp(P_i)/D$ for $i = 1, \ldots, 3$; $\phi_4 = 1/D$; $\sigma_\varepsilon = (SE^2)^{1/2}$; $\bar{\lambda} = (L^2)$; $\alpha_i = [(S_i)^2]^{1/2}$ for $i = 1, \ldots, 4$; and $D = 1 + \exp(P_1) + \exp(P_2) + \exp(P_3)$. The free parameters in the DFP algorithm are then taken to be H, B, T, S, P_1, P_2, P_3, SE, S_1, S_2, S_3, S_4, γ, ν, and β_1. Similarly, the autoregressive polynomials in $\overline{H}(L)$ are constrained to have roots outside the unit circle. The implied transformations for the coefficients $\bar{a}_{ij}$ for $i = 1, \ldots, 4$ and $j = 1, 2$ are given by $\bar{a}_{i1} = \rho_i[\cos\theta_i - \sin\theta_i]/2$ and $\bar{a}_{i2} = (\rho_i[\cos\theta_i + \sin\theta_i]/2) - 1$. In these expressions, $\theta_i = (\pi/2)/[1 + \exp(x_i)]$, $R_i = 4/(\cos^2\theta_i + \sin^2\theta_i)$, $\rho_i = R_i/[1 + \exp(y_i)]$, and x_i, y_i, b_i for $i = 1, \ldots, 4$ constitute the free parameters.

8/The estimates of the variance σ_ε^2 of the innovation to the technology shock are not invariant to the fact that the approximation around the steady states has been performed by normalizing total leisure hours as unity. The estimates of σ_ε^2 in Table 3 are obtained under the assumption that leisure hours in a quarter total 2,142.

9/The unrestricted spectral estimates are computed as the Daniell smoothed periodogram estimates. The window width for the smoothed estimates is chosen to be 11. With that window width, the asymptotic distribution of the unrestricted spectral density ordinates is proportional to a χ^2 random variable with 11 degrees of freedom. Thus, the confidence band for the logarithm of the unrestricted estimates is defined as $[\log(11/a) + \log(f(\omega)), \log(11/b) + \log(f(\omega))]$, where $f(\omega)$ is the Daniell estimator and a and b are defined as $\Pr(\chi^2_{11} < a) = 0.025$ and $\Pr(\chi^2_{11} < b) = 0.095$, that is, $a = 3.81575$ and $b = 26.7569$. (See Koopmans [1974], pp. 268-70, 274-77.) All calculations use the Regression Analysis of Time Series (RATS) package, version 4, copyright by VAR Econometrics, 1980.

10/To see how this filter is derived, recall that Kydland and Prescott studied the behavior of the deviations of each series, y_t, from some smooth series s_t. This series was, in turn, generated by minimizing $\sum_{t=1}^{T}(y_t - s_t)^2 + 1600\sum_{t=1}^{T}(s_{t+1} - 2s_t + s_{t-1})^2$ with respect to s_t. Denote the solution $s_t = k(L)y_t$, where $k(L)$ is a two-sided lag polynomial. The series for the deviations, denoted y_t', is then obtained as $y_t' = y_t - [1/k(L)]y_t$. Hence, the filter applied to the data is $1 - [1/k(L)] = 1600(L-1)^2(L^{-1})^2/(L-\lambda_1)(L-\lambda_2)(L^{-1}-\lambda_3)(L^{-1}-\lambda_4)$, where λ_1 and λ_2 are less than one, λ_3 and λ_4 are greater than one in modulus, and L is the lag operator.

Figures 1-4

Log Spectra of the Four Series

KEY: Estimates Restricted by (3.1) = R
(3.2) = F

Unrestricted Estimate = U

95% Confidence Interval for U = *

Figure 1 Output

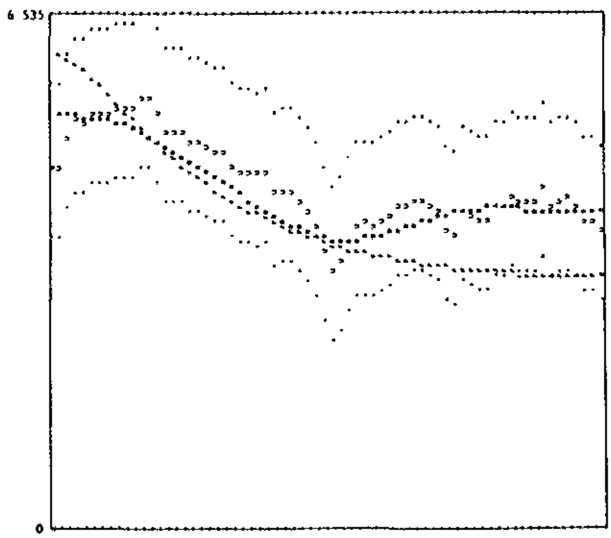

Figure 2 Hours

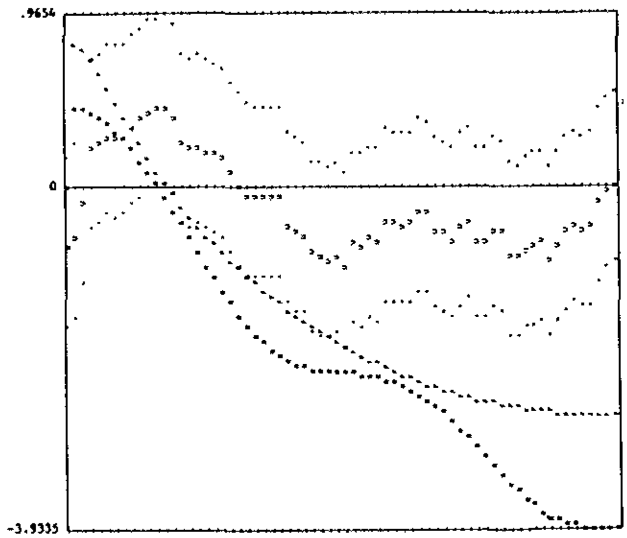

Figure 3 Structures

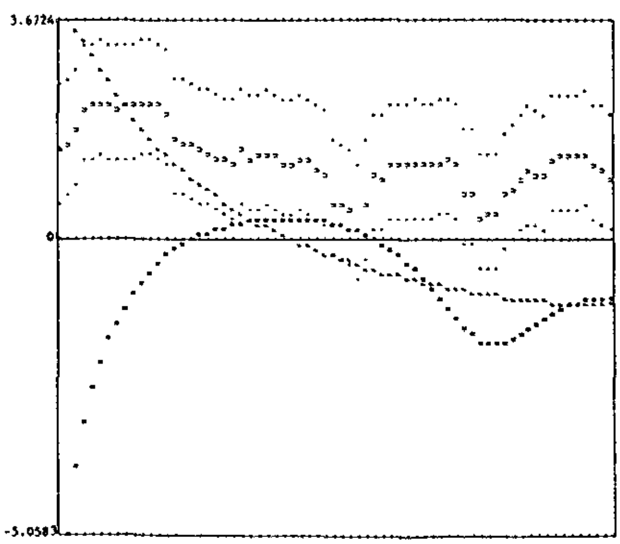

Figure 4 Equipment

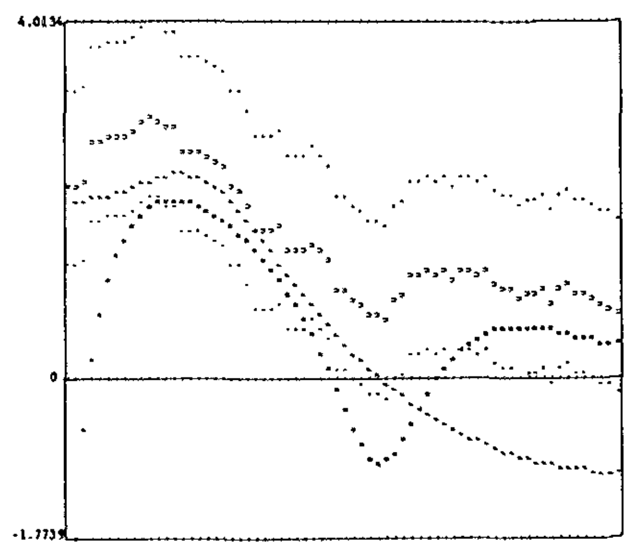

Table 1

Tests of Model (5.1)

Frequency	Period (Quarters)	Ordinates	One-Noise Null Tests: χ^2-Statistic*	One-Noise Null Tests: Marginal Significance Level**	Coherences‡: Output	Coherences‡: Hours	Coherences‡: Investment: Structures	Coherences‡: Investment: Equipment
$.0694\pi$	28.8	2-10	1.9027	.86244	.88044	.83923	.67441	.84179
$.2361\pi$	8.5	14-22	3.7058	.59250	.85985	.76693	.66661	.79371
$.4028\pi$	4.9	26-34	5.0480	.41005	.63518	.58511	.56428	.53929
$.5694\pi$	3.5	38-46	1.4312	.92087	1.00000	.44323	.19990	.24556
$.7361\pi$	2.7	50-58	3.9150	.56172	.19110	.30936	.03416	1.00000
$.9028\pi$	2.2	62-70	7.9202	.16069	.62612	.41179	.62885	.58378
Overall	--	--	23.9229	--	.70121	.57483	.49484	.62695

*For frequencies, $\chi^2(5)$; overall, $\chi^2(30)$.

**Defined as $\Pr(\chi^2(df) > c)$, where c is the value of the test statistic and df the number of degrees of freedom.

‡Each row shows the coherence of the four series with the unobservable factor at the frequency listed in the first column. The overall coherence indicates the total proportion of the variance explained over all six bands.

Table 2

Tests of Serial Correlation

Table 2a. Estimates of $\hat{V}_i(\omega)$ for i = 1, ..., 4

	Series			
Frequency	Output	Hours	Structures	Equipment
$.0694\pi$	70.561	1.1171	10.464	5.6095
$.2361\pi$	90.783	1.8353	15.425	4.4404
$.4028\pi$	150.440	1.2805	14.010	11.2150
$.5694\pi$	.000	3.1726	19.350	25.1130
$.7361\pi$	438.560	3.4469	30.328	.0001
$.9028\pi$	246.220	4.7146	15.489	15.6870

Table 2b. Wald Test for No Serial Correlation
in $\{\tilde{v}_{it}\}_{t=0}^{\infty}$ for i = 1, ..., 4

Series	$\chi^2(5)$	Marginal Significance Level
Output	.987	.97020
Hours	1.080	.96250
Structures	1.570	.91307
Equipment	12.270	.03470

Table 3

Estimates of Model (3.1)

Parameters and Shock Variances	(a) $l_t^* = 1 - (1-\eta)n_t - \eta(1-\eta)a_t$	(b) $l_t^* = l_t + \beta_1 l_{t-1}$
Preferences		
γ	1.6589 (.658)	1.0000
η	.00085 (.00086)	---
β_1	---	.8064 (.130)
b	.3142 (.152)	.2530 (.038)
Technology		
ν	.4492 (.632)	1.4407 (.441)
θ	.9884 (.247)	.9812 (.044)
σ	.0802 (.072)	.0194 (.016)
ϕ_1	.1106 (.062)	.1340 (.051)
ϕ_2	.2736 (.042)	.2697 (.018)
ϕ_3	.2832 (.042)	.2899 (.015)
ϕ_4	.3326 (.042)	.3063 (.045)
$\bar{\lambda}$	3.5773 (1.871)	5.5979 (.441)
Shocks		
σ_ε^2	.00105 (.00003)	.00088 (.00003)
σ_1^2	20.3122 (7.388)	14.7355 (3.600)
σ_2^2	3.6186 (1.090)	3.8163 (.694)
σ_3^2	27.3104 (6.194)	28.6823 (4.040)
σ_4^2	79.9229 (35.485)	186.2847 (66.154)
Likelihood function $L_T^*(\underline{\theta})$	-1,832.2423	-1,832.5860
$\sum_{j=1}^{T} \log\,[\det S_\xi(\omega_j;\underline{\theta})]$	-948.8569	-944.7600
$\sum_{j=1}^{T} \text{trace}\,[S_\xi(\omega_j;\underline{\theta})^{-1} I(\omega_j)]$	-267.7018	-272.1369

Note: The numbers in parentheses are standard errors.

Table 4

Estimates of Model (3.2)

i	x_i	y_i	b_i	σ_i^2
1	-1.849 (.211)	-.825 (.307)	4.360 (.329)	11.6008 (1.439)
2	-2.541 (.470)	-.135 (.300)	1.778 (.096)	1.6363 (.364)
3	-2.811 (.509)	-.081 (.278)	3.504 (.251)	7.6102 (1.000)
4	-1.715 (.365)	-.141 (.229)	12.156 (1.609)	345.9764 (12.001)

Likelihood function $L_T^*(\underline{\theta}) = -1{,}784.3374$

$\sum_{j=1}^{T} \log\,[\det S_\xi(\omega_j;\underline{\theta})] = -898.5634$

$\sum_{j=1}^{T} \operatorname{trace}\,[S_\xi(\omega_j;\underline{\theta})^{-1} I(\omega_j)] = -270.0851$

Note: The numbers in parentheses are standard errors.

Table 5

Impulse Responses From Models (3.1) and (3.2)

Shocked Series and Model	Period (Quarters)	Response of Series: Output	Hours	Investment: Structures	Investment: Equipment
y_t from (3.1)	1	20.4734	.4129	1.2406	.02126
	2	20.5473	.4180	1.2201	.02279
	3	20.5497	.4084	1.1904	.02238
	4	20.5475	.4082	1.1895	.02230
	5	20.5413	.4516	1.3282	.02461
	6	20.5928	.4531	.7339	.00420
	7	20.5943	.4512	.7371	.00368
	8	20.5930	.4510	.7506	.00416
	9	20.5922	.4560	.7674	.00441
	10	20.5980	.4562	.7000	.00209
	11	20.5983	.4559	.6986	.00198
	12	20.5981	.4559	.7015	.00207
	13	20.5979	.4565	.7036	.00212
	14	20.5986	.4565	.6959	.00185
	15	20.5987	.4564	.6956	.00183
$\bar{y}_t$ from (3.2)	1	18.6023	1.2792	3.4059	2.7586
	2	29.1045	2.3622	6.4537	5.1579
	3	36.3417	3.1932	7.8508	7.1330
	4	41.1660	3.8241	7.9105	8.7538
	5	44.3986	4.3025	7.4183	10.0834
	6	46.5628	4.6651	6.9545	11.1743
	7	48.0119	4.9400	6.7317	12.0692
	8	48.9822	5.1484	6.7135	12.8034
	9	49.6319	5.3063	6.7842	13.4057
	10	50.0669	5.4261	6.8546	13.8999
	11	50.3582	5.5169	6.8900	14.3053
	12	50.5532	5.5857	6.8941	14.6379
	13	50.6838	5.6378	6.8840	14.9107
	14	50.7713	5.6774	6.8734	15.1346
	15	50.8298	5.7074	6.8678	15.3182

Table 6

The Proportion of Variance in the Four Series Explained by One Common Factor:
Coherences for ξ_t

		Table 6a. From Model (3.1)				Table 6b. From Model (3.2)			
				Investment				Investment	
Frequency	Period (Quarters)	Output	Hours	Structures	Equipment	Output	Hours	Structures	Equipment
0.0000	.00	.93758	.80809	.04986	.00146	.94546	.91192	.71349	.95268
0.0149π	134.00	.93751	.80460	.15339	.00458	.94344	.90525	.71522	.94556
0.0299π	67.00	.93634	.79863	.27697	.00987	.94010	.89433	.71805	.93394
0.0448π	44.66	.93699	.79001	.39042	.01742	.93548	.87948	.72191	.91815
0.0598π	33.50	.93637	.77849	.48225	.02720	.92965	.86107	.72669	.89863
0.0744π	26.80	.93540	.76383	.55203	.03904	.92267	.83958	.73223	.87590
0.0894π	22.33	.93540	.74580	.60329	.05261	.91463	.81550	.73836	.85053
0.1044π	19.14	.93214	.72422	.64004	.06746	.90562	.78935	.74482	.82306
0.1193π	16.75	.92976	.69905	.66573	.08305	.89574	.76164	.75133	.79407
0.1343π	14.88	.92687	.67036	.68294	.09888	.88509	.73285	.75755	.76406
0.1492π	13.40	.92345	.63839	.69362	.11449	.87378	.70343	.76370	.73351
0.1642π	12.18	.91952	.60357	.69913	.12955	.86191	.67378	.76753	.70284
0.1792π	11.16	.91508	.56649	.70049	.14381	.84960	.64423	.77039	.67239
0.1941π	10.30	.91015	.52788	.69840	.15713	.83694	.61507	.77120	.64245
0.2088π	9.57	.90474	.48859	.69335	.16944	.82404	.58654	.76950	.63126
0.2536π	7.88	.88579	.37524	.66337	.20030	.78469	.50636	.74559	.53170
0.3284π	6.09	.84566	.23908	.56890	.23308	.72121	.39512	.63899	.41951
0.4029π	4.96	.79853	.18246	.41439	.24384	.66528	.31171	.48852	.33581
0.4774π	4.18	.76040	.17063	.21863	.22944	.61932	.25108	.35383	.27493
0.5522π	3.62	.75655	.16362	.10910	.18522	.58347	.20756	.25728	.23102
0.6267π	3.19	.78264	.13898	.18343	.11713	.55676	.17655	.19408	.19951
0.7314π	2.79	.80409	.10664	.28032	.06557	.53775	.15472	.15394	.17716
0.7910π	2.52	.81801	.06641	.34807	.03611	.52494	.13978	.12891	.16174
0.8658π	2.31	.81914	.04223	.35374	.04640	.51700	.13022	.11401	.15181
0.9402π	2.12	.81489	.03408	.33125	.06781	.51289	.12510	.10641	.14648

Appendix A

The Approximate Equilibrium Laws of Motion

This appendix derives the approximate equilibrium laws of motion defined by equations (2.14) and (2.15). These laws of motion are, with a small modification, consistent with two different specifications for the technology shock process. Under the first specification, the technology shock is defined as the sum of a random walk plus a process always constant--that is, $\lambda_t = \lambda_{1t} + \lambda_{2t}$, where $\lambda_{1t} = \lambda_{1t-1} + \varepsilon_t$ and $\lambda_{2t} = \bar{\lambda}$. Under the second, the technology shock follows a random walk with drift--that is, $\lambda_t = \lambda_{t-1} + \varepsilon_t + \bar{\lambda}$. In both cases, the single-period utility function in the social planner's problem defined by equation (2.13) is approximated by a quadratic function. To obtain the point around which this approximation is performed, the technology shock is set equal to its unconditional mean of zero. Hence, the approximation is performed around the point to which the original economy converges in the absence of random shocks to the technology process.

Under the first specification for the $\{\lambda_t\}_{t=0}^{\infty}$ process, such a point will always exist, but it cannot be interpreted as a deterministic steady state for the levels of the different series. This is because that point will depend on the initial condition λ_{10} for the $\{\lambda_{1t}\}_{t=0}^{\infty}$ process, as well as on the parameter $\bar{\lambda}$. Because this initial condition is unobservable, however, it can be normalized at a value of zero without affecting the estimates of any of the remaining parameters. Under the second specification for the technology shock process, such a point will constitute a deterministic steady state for the differences of each variable. In this case, provided the quadratic approximation is performed with respect to the differences of each variable, the point around which this approximation is performed will be independent of initial conditions for λ_t. In the former case, the methods described in this appendix yield representations for the levels of each variable. These representations are then differenced to eliminate the unit root in the technology process, leading to the stationary stochastic processes for the differences described by equations (2.14) and (2.15). In the latter case, the approximate equilibrium laws of motion are derived directly for the differences. Because of the certainty equivalence property exhibited by the solution procedure, however, the only difference between the representations derived directly for the differences and those displayed in equations (2.14) and (2.15) occurs in the B(L) matrix polynomial defined by equation (A.7). Specifically, the first term in the braces following the summation sign is not divided by $1 - \mu_j\beta$ for $j = 1, \ldots, J + 1$ when the solution is computed for the differences directly. This occurs because the laws of motion for the differences of each variable depend on $\sum_{i=0}^{\infty}(\mu_j\beta)^i E(\lambda_{t+i}-\lambda_{t+i-1}|\Omega_t) = \varepsilon_t$ while the representations derived by differencing the laws of motion for the levels depend on $\sum_{i=0}^{\infty}(\mu_j\beta)^i E(\lambda_{t+i}|\Omega_t) - \sum_{i=0}^{\infty}(\mu_j\beta)^i E(\lambda_{t+i-1}|\Omega_{t-1}) = (1-\mu_j\beta)^{-1}\varepsilon_t$. If μ_j is small, however, $(1-\mu_j\beta)^{-1}$ will be close to one and the two approaches will yield very similar results for the differences of each variable.

1. The Deterministic Equilibrium

This section defines the deterministic equilibrium of the original economy in terms of the constants k_1, k_2, ... These expressions may be interpreted in terms of the levels or the differences of each series. Under the first interpretation, they depend on λ_{10}, but this dependence has been suppressed for notational simplicity.

In the deterministic equilibrium, the interest rate and the shadow price of the first type of capital are given by

$$r = (1-\beta)/\beta \quad \text{and} \quad q = \sum_{j=1}^{J} (1+r)^{j-J}\phi_j$$

and the costs of the two types of capital are $q(r+\delta_1)$ and $r + \delta_2$. The values of the capital stocks are obtained by equating the marginal product of each type to its cost. Hence, $f_{k_1} = q(r+\delta_1)$ and $f_{k_2} = r + \delta_2$ and

$$k_2 = \left[\frac{(r+\delta_1)q}{r+\delta_2}\left(\frac{\sigma}{1-\sigma}\right)\right]^{1/(1+\nu)}(k_1) = b_1k_1$$

$$k_1 = \left[\frac{(1-\theta)(1-\sigma)}{q(r+\delta_1)}\, b_2^{-(1-\theta+\nu)/\nu}\right]^{1/\theta}(\bar{\lambda}^{1/\theta})n = b_3(\bar{\lambda}^{1/\theta})n$$

where $b_2 = 1 - \sigma + \sigma b_1^{-\nu}$. In this equilibrium, both types of investment are a constant fraction of the relevant capital stocks, that is, $i_1 = \delta_1k_1$ and $i_2 = \delta_2k_2$ while the production function and the national income identity provide the relevant values for output Q and consumption c as

$$Q = b_2^{-(1-\theta)/\nu}(b_3^{1-\theta})(\bar{\lambda}^{1/\theta})n = b_4(\bar{\lambda}^{1/\theta})n$$

$$c = b_4(\bar{\lambda}^{1/\theta})n - \delta_1k_1 - \delta_2k_2 = (b_4-\delta_1b_3-\delta_2b_1b_3)(\bar{\lambda}^{1/\theta})n.$$

Finally, the value of hours is obtained by setting the marginal rate of substitution of consumption for leisure equal to the marginal product of labor. When $l_t^* = 1 - (1-\eta)n_t - \eta(1-\eta)a_t$,

$$n = \left[1 + \frac{(1-b)(1-\eta)(b_4-\delta_1b_3-\delta_2b_1b_3)}{bb_4\theta(1-\beta\eta)}\right]^{-1} \text{and } a = \frac{\eta}{1-\eta}.$$

When $l_t^* = l_t + \beta_1 l_{t-1}$,

$$n = \left[1 + \frac{(1-b)(1+\beta_1\beta)(b_4-\delta_1b_3-\delta_2b_1b_3)}{bb_4\theta(1+\beta_1)}\right]^{-1}.$$

2. The Approximate Social Planner's Problem

Once the deterministic equilibrium values of all variables are computed as functions of the underlying parameters of the model, the single-period utility function in (2.13) is approximated by a quadratic function according to the procedure described by Kydland and Prescott [1982, pps. 1356-57]. In the approximation procedure, the percentage deviations around the deterministic equilibrium values of k_{1t}, a_t, $i_{1t} + i_{2t}$, k_{2t}, n_t, and λ_t are taken to be 1, 2, 8, 3, 3, and 0.5 percent. While these deviations are set somewhat arbitrarily, changing their values has very little effect on the results of the estimation. Now the constraints of (2.13) are used to eliminate i_{1t}, i_{2t}, and n_t from the quadratic objective function. Then the approximate social planner's problem for the economy of section 2 is described by

$$\text{(A.1)} \qquad \underset{\{x_t\}}{\text{maximize}}\ E\left(\sum_{t=0}^{\infty}\beta^t[b_*'(D(L)x_t-s) + (D(L)x_t-s)'Q_1(D(L)x_t-s) + 2q_*'(D(L)x_t-s)(\lambda_t-\bar{\lambda}) + q_6(\lambda_t-\bar{\lambda})^2 + b_6(\lambda_t-\bar{\lambda})]\,|\,\Omega_0\right)$$

given the initial conditions $x_{-J}, \ldots, x_{-1}$. Here $x_t = (k_{1t+J}, a_{t+1}, k_{2t+1})'$ is the vector of decision variables, $s = (k_1, a, i_1 + i_2, k_2, n)'$ is the vector of deterministic equilibrium values, $D(L)$ is a J^{th}-order matrix polynomial given by

$$D(L) = \begin{bmatrix} L^J & 0 & 0 \\ 0 & L & 0 \\ g(L) & 0 & 1-(1-\delta_2)L \\ 0 & 0 & L \\ 0 & 1-\eta L & 0 \end{bmatrix}$$

and the remaining terms are defined from the linear and quadratic coefficients $\bar{b}$ and Q of the approximate utility function as

$$\bar{b} = [b_* | b_6] \quad \text{and} \quad Q = \left[\begin{array}{c|c} Q_1 & q_* \\ \hline q_* & q_6 \end{array}\right] .$$

3. Solution to the Approximate Social Planner's Problem

Following Hansen and Sargent [1981], the necessary conditions for the maximization of (A.1) consist of the set of Euler equations

$$\text{(A.2)} \qquad D(\beta L^{-1})'Q_1 D(L)x_t = D(\beta L^{-1})'q_*\lambda_t - (1/2)D(\beta L^{-1})'(b_*-2Q_1 s-2\bar{\lambda})$$

for t = 0, 1, 2, ... and a set of transversality conditions which require that the optimal path for x_t or, alternatively, $x_t^* = (k_{1t},a_t,k_{2t})'$ satisfy $\sum_{t=0}^{\infty}\{\beta^t x_t^{*}{}'Gx_t^*\} < \infty$, where G is a positive-definite matrix obtained by suitably partitioning $(D(L)x_t)'Q_1(D(L)x_t)$.

Simplifying the terms on the right side of (A.2) yields

$$\text{(A.3)} \qquad D(\beta L^{-1})'Q_1 D(L)x_t = A_0\lambda_t + A_1\lambda_{t+1} + \cdots + A_J\lambda'_{t+J} + h$$

for t = 0, 1, 2, ... In this expression, A_k for k = 0, ..., J and h are obtained from $D(\beta L^{-1})'q_*\lambda_t$ and the last term in (A.2), using the definition of D(L). The elements of A_k for k = 0, ..., J are defined as a_{ik} for i = 1, 2, 3. $D(\beta z^{-1})'Q_1 D(z)$ can be factored as $C(\beta z^{-1})'C(z)$, where C(z) is a J^{th}-order matrix polynomial in nonnegative powers of z. This factorization insures that the roots of det C(z) = 0 are greater than $\beta^{1/2}$ in modulus and the roots of det $C(\beta z^{-1})' = 0$ are less than $\beta^{-1/2}$. To determine the number of nonzero roots of det C(z) = 0, notice that Q_1 is a positive-definite symmetric matrix by construction and may be decomposed as $Q_1 = PP'$. More precisely,

$D(\beta z^{-1})'Q_1D(z) = D(\beta z^{-1})'PP'D(z) = C(\beta z^{-1})'C(z)$. Hence, the number of nonzero roots of det $C(z) = 0$ are found from the number of nonzero roots of the principal minors of $P'D(z)$. It is tedious but straightforward to show that the principal minors of $P'D(z)$ have $J + 1$ nonzero roots with the remaining root at $z = 0$. In this case, det $C(z) = 0$ also has $J + 1$ nonzero roots.

The solution to the set of difference equations in (A.3) is obtained by solving the stable roots backward and the unstable roots forward. For this purpose, the feedback polynomial $C(L)$ is obtained by means of a numerical factorization routine. Then $[C(\beta L^{-1})']^{-1}$ is evaluated by using the identity $[C(\beta z^{-1})']^{-1} = \text{adj } C(\beta z^{-1})'/\det C(\beta z^{-1})'$. Since the roots of det $C(\beta z^{-1})' = 0$ equal the inverse of $\beta^{1/2}$ times the roots of det $C(z) = 0$, the zero root of det $C(z) = 0$ corresponds to a root at infinity of det $C(\beta z^{-1})' = 0$. When this occurs, the matrix partial fractions decomposition of $[C(\beta z^{-1})']^{-1}$ is defined by

$$\text{(A.4)} \qquad \begin{aligned}[C(\beta z^{-1})']^{-1} &= \text{adj } C(\beta z^{-1})'/\det C(\beta z^{-1})' \\ &= \sum_{j=1}^{J+1} -(1/z_j)N_j/[1-(1/z_j)\beta z^{-1}] + N_{J+2} \\ &= \sum_{j=1}^{J+1} \mu_j N_j/(1-\mu_j \beta z^{-1}) + N_{J+2}.\end{aligned}$$

In this expression, N_{J+2} corresponds to the zero root of det $C(z) = 0$ and $\mu_j = z_j^{-1}$ for $j = 1, \ldots, J - 1$. (See Nerlove, Grether, and Carvalho [1979, p. 369].) If the service technology for leisure is defined by (2.5) or, alternatively, $\eta = 0$ in the distributed lag polynomial (2.2), then det $C(z) = 0$ will have only J nonzero roots and the formula in (A.4) is modified accordingly. In any case, the matrices N_j for $j = 1, \ldots, J + 2$ are found according to the recursive formulas in Hansen and Sargent [1981].

These results can be used to express (A.3) as

$$\text{(A.5)}\qquad C(L)x_t = \sum_{j=1}^{J+1}(\mu_j N_j)\Big[\sum_{i=0}^{\infty}(\mu_j\beta)^i h + A_0\sum_{i=0}^{\infty}(\mu_j\beta)^i\lambda_{t+i} + \cdots$$

$$+ A_J\sum_{i=0}^{\infty}(\mu_j\beta)^i\lambda_{t+i+J}\Big] - N_{J+2}\sum_{k=0}^{J}A_k\lambda_{t+k} - N_{J+2}h.$$

(A.6) is obtained from (A.5) by replacing λ_{t+i} for $i > 0$ with its forecast conditional on the appropriate information set. For the capital stock decisions, the relevant information set is $\Omega_t = \{\lambda_t, \lambda_{t-1}, \ldots\}$, while for the labor/leisure allocation, it is $\Omega_{t-1} = \{\lambda_{t-1}, \lambda_{t-2}, \ldots\}$. Then

$$\text{(A.6)}\qquad C(L)x_t = \Big[\sum_{j=1}^{J+1}\frac{\mu_j N_j}{1-\mu_j\beta} - N_{J+2}\Big]\Big[h + \bar{\lambda}\sum_{k=0}^{J}A_k\Big]$$

$$+ \left\{\sum_{j=1}^{J+1}\frac{\mu_j N_j}{1-\mu_j\beta}\begin{bmatrix} a_{01}+\sum_{k=1}^{J}a_{k1} \\ 0 \\ a_{03}+a_{13}\end{bmatrix} - N_{J+2}\begin{bmatrix} a_{01}+\sum_{k=1}^{J}a_{k1} \\ 0 \\ a_{03}+a_{13}\end{bmatrix}\right\}\lambda_{1t}$$

$$+ \left\{\sum_{j=1}^{J+1}\frac{\mu_j N_j}{1-\mu_j\beta}\begin{bmatrix} 0 \\ a_{02}+a_{12} \\ 0\end{bmatrix} - N_{J+2}\begin{bmatrix} 0 \\ a_{02}+a_{12} \\ 0\end{bmatrix}\right\}\lambda_{1t-1}.$$

Letting I denote the 3 x 3 identity matrix and applying $(1-L)I$ to both sides of (A.6) yields

$$(\text{A.7})\quad (1-L)IC(L)x_t = \left\{ \sum_{j=1}^{J+1} \frac{\mu_j N_j}{1-\mu_j\beta} \begin{bmatrix} a_{01}+\sum_{k=1}^{J} a_{k1} \\ 0 \\ a_{03}+a_{13} \end{bmatrix} - N_{J+2} \begin{bmatrix} a_{01}+\sum_{k=1}^{J} a_{k1} \\ 0 \\ a_{03}+a_{13} \end{bmatrix} \right\} \varepsilon_t$$

$$+ \left\{ \sum_{j=1}^{J+1} \frac{\mu_j N_j}{1-\mu_j\beta} \begin{bmatrix} 0 \\ a_{02}+a_{12} \\ 0 \end{bmatrix} - N_{J+2} \begin{bmatrix} 0 \\ a_{02}+a_{12} \\ 0 \end{bmatrix} \right\} \varepsilon_{t-1}$$

$$(1-L)Ix_t = C(L)^{-1}(B_0+B_1L)\varepsilon_t = C(L)^{-1}B(L)\varepsilon_t.$$

Given (A.7), the equilibrium laws of motion for the remaining quantity variables are derived by linearizing the CES production function for output around the deterministic equilibrium values of n_t, k_{1t}, k_{2t}, and $\lambda_t = \bar{\lambda}$ as

$$Q_t = a_1(k_{1t}-k_1) + a_2(n_t-n) + a_3(k_{2t}-k_2) + a_4(\lambda_t-\bar{\lambda})$$

$$= a_1 i_{1t} + a_2 n_t + a_3 k_{2t} + a_4 \lambda_t + \bar{Q}.$$

Define the matrix polynomial $G = (0,0,0,a_4)'$, the vector $y_t = (i_{1t}, n_t, i_{2t}, Q_t)'$, and apply the transformation

$$E(L) = \begin{bmatrix} g(L) & 0 & 0 \\ 0 & 1-\eta L & 0 \\ 0 & 0 & 1-(1-\delta_2)L \\ a_1 L^J & a_2(1-\eta L) & a_3 L \end{bmatrix}$$

to both sides of (A.7). This yields $(1-L)Iy_t = [E(L)C(L)^{-1}B(L)+G]\varepsilon_t$.

Appendix B

The Approximate Likelihood Function

To derive the approximate likelihood function, let $\underline{\xi} = (\xi_1, \ldots, \xi_T)'$ denote a 4T vector containing a sample of observations on the stationary stochastic process $\{\xi_t\}_{t=0}^{\infty}$ for $t = 1, \ldots, T$ and define $\Gamma(\theta) = E(\underline{\xi}\,\underline{\xi}')$ to be the variance-covariance matrix of $\underline{\xi}$ as a function of some parameter vector θ. Then the log-likelihood function for $\underline{\xi}$ can be expressed as

$$\text{(B.1)} \qquad L_T(\theta) = -\frac{1}{2}(T{+}4T)\log 2\pi - \frac{1}{2}\log[\det \Gamma(\theta)] - \frac{1}{2}\underline{\xi}'\Gamma(\theta)\underline{\xi}.$$

From the results derived by Hannan [1970, p. 378] and recently suggested by Hansen and Sargent [1981], the frequency domain approximation to $L_T(\theta)$, denoted $L_T^*(\theta)$, is given by

$$\text{(B.2)} \qquad L_T^*(\theta) = -\frac{1}{2}(T{+}4T)\log 2\pi - \frac{1}{2}\sum_{j=0}^{T-1}\log[\det S_\xi(\omega_j;\theta)]$$

$$-\frac{1}{2}\sum_{j=1}^{T-1}\text{trace}\,[S_\xi(\omega_j;\theta)^{-1}I(\omega_j)].$$

Here $\omega_j = 2\pi j/T$ for $j = 1, \ldots, T-1$ are the harmonic frequencies, $I(\omega_j)$ for $j = 1, \ldots, T-1$ is the periodogram for the sample of observations $\{\xi_t\}_{t=1}^{T}$, and $S_\xi(\omega_j;\theta)$ is the spectral density matrix of the $\{\xi_t\}_{t=0}^{\infty}$ process at frequency ω_j obtained by evaluating the covariance generating function of $\{\xi_t\}_{t=0}^{\infty}$ at $z = e^{i\omega_j}$. The periodogram summarizes the covariance properties of the sample of observations in the approximate likelihood function. Its ordinate at frequency ω_j is defined by

$$\text{(B.3)} \qquad I(\omega_j) = \frac{1}{T}\left[\sum_{j=1}^{T}\xi_t e^{it\omega_j}\right]\left[\sum_{j=1}^{T}\xi_t' e^{-it\omega_j}\right].$$

The spectral density matrix represents the restrictions of the underlying economic model for the covariances of the $\{\xi_t\}_{t=0}^{\infty}$ process at each frequency. If ξ_t evolves according to (3.1), the spectral density matrix $S_\xi(\omega_j;\theta)$ at frequency ω_j is defined by

$$\text{(B.4)} \qquad \begin{aligned} S_\xi(\omega_j;\underline{\theta}) &= \frac{1}{2\pi}[\sigma_\varepsilon^2 H(\omega_j)H(\omega_j)^* + V] \\ &= \frac{1}{2\pi}\{\sigma_\varepsilon^2[E(\omega_j)C(\omega_j)^{-1}B(\omega_j) + G] \\ &\qquad \times [E(\omega_j)C(\omega_j)^{-1}B(\omega_j) + G]^* + V\}. \end{aligned}$$

If the correct model for ξ_t is (3.2), then $S_\xi(\omega_j;\theta)$ is defined by

$$\text{(B.5)} \qquad S_\xi(\omega_j;\overline{\underline{\theta}}) = \frac{1}{2\pi}\{\overline{H}(\omega_j)\overline{H}(\omega_j)^* + \overline{V}\}.$$

In equations (B.4) and (B.5), an asterisk (*) denotes both conjugation and transposition, $V = E(v_t v_t')$, and $\overline{V} = E(\bar{v}_t \bar{v}_t')$. Also, $H(\omega_j)$, $E(\omega_j)$, $C(\omega_j)$, $B(\omega_j)$, and $\overline{H}(\omega_j)$ are the Fourier transforms of $H(L)$, $E(L)$, $C(L)$, $B(L)$, and $\overline{H}(L)$, respectively.

Subject to some regularity conditions, the results in Hannan [1970] can be used to show that the estimates obtained by maximizing (B.3) with respect to the vector of parameters θ are strongly consistent, asymptotically normal, and efficient and that the Wald and likelihood ratio test statistics have the usual distributions. Furthermore, these results hold even if the underlying distribution generating the observations is not normal.

To find the value of the approximate likelihood function for a given vector of parameters θ, $I(\omega_j)$ and $S_\xi(\omega_j;\theta)$ must be computed for $j = 1, \ldots, T-1$. Since the periodogram does not depend on θ, it is computed only once and that value is used at every subsequent evaluation of the likelihood function. Calculating the value of $S_\xi(\omega_j;\theta)$ under representation (B.5) is also simple and requires finding the Fourier transform of $\overline{H}(L)$ for each set of parameters $\theta = \overline{\underline{\theta}}$. Determining the value of $S_{\xi}(\omega_j;\theta)$ when it is defined by equation (B.5) requires more work. This is because the mapping from the vector of parameters $\underline{\theta}$ to $H(L)$ is not explicit. The expressions in Appendix A for the deterministic equilibrium values, the linear and quadratic coefficients $\bar{b}$ and Q, and the lag polynomial $D(L)$ yield all the terms in the approximate social planner's problem in terms of the given set of parameter values $\underline{\theta}$. However, the feedback polynomial $C(L)$ is obtained by a numerical factorization routine while the inverse of the feed-forward polynomial $C(\beta L^{-1})$ is evaluated numerically using a recursive procedure. Once $C(L)$ and $C(\beta L^{-1})^{-1}$ are determined, however, explicit formulas for $E(L)$, G, and the matrices A_k for $k = 0, \ldots, J$ yield the value of $H(L)$ at $\underline{\theta}$.

Appendix C

The Impulse Response Functions

This appendix describes the representations used to calculate the impulse responses of Table 5. The representation used to derive the results from model (3.1) is

$$y_t - y_{t-1} = [E(L)C(L)^{-1}B(L) + G]\varepsilon_t. \tag{C.1}$$

The lag polynomials E(L), C(L), and B(L) and the constant matrix G in this equation are evaluated at the parameter values $\beta = 0.9909$, $\delta_1 = 0.02$, $\delta_2 = 0.03$, $\gamma = 1$, $\eta = 0$, $b = 0.3342$, $\theta = 0.9788$, $\nu = 1.0377$, $\sigma = 0.0254$, $\phi_1 = 0.146$, $\phi_2 = 0.237$, $\phi_3 = 0.3034$, $\phi_4 = 0.3134$, and $\bar{\lambda} = 2.085$. Their implied values are given by

$$E(L) = \begin{bmatrix} .3134 - .003716L - .0639L^2 - .08618L^3 - .14315L^4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-.97L \\ .03055L^J & .0396 & 2.831 \end{bmatrix}$$

$$G = (0,0,0,.3230)'$$

$$C(L) = C_0 + C_1L + C_2L^2 + C_3L^3 + C_4L^4 \text{ with } C_0 = I$$

$$C_1 = \begin{bmatrix} -.016101 & 0 & .173210 \\ -.003609 & 0 & -.283230 \\ -.002440 & 0 & .036925 \end{bmatrix} \qquad C_2 = \begin{bmatrix} -.021701 & 0 & 0 \\ -.021989 & 0 & 0 \\ -.001076 & 0 & 0 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} -.002112 & 0 & 0 \\ -.014881 & 0 & 0 \\ .040640 & 0 & 0 \end{bmatrix} \qquad C_4 = \begin{bmatrix} .106409 & 0 & 0 \\ -.028200 & 0 & 0 \\ .022683 & 0 & 0 \end{bmatrix}$$

$B(L) = B_0 + B_1 L$ with $B_0 = (0.18081, 0.02346, 0.01617)'$ and $B_1 = (0,0,0)'$. The (inverse) roots of det $C(z) = 0$ are $\mu_1 = 0.57525$, $\mu_2 = -0.567394$, $\mu_3 = (0.0064822, -0.58972)$, and $\mu_4 = (0.0064822, 0.58972)$. Similarly, the roots of $g(z) = 0$ are $\lambda_1 = 1.0204$, $\lambda_2 = -1.2965$, $\lambda_3 = (-0.1629, 1.276)$, and $\lambda_4 = (-0.1629, -1.276)$.

The impulse responses from model (3.2) are computed from the elements $(1-\bar{a}_{i1}L-\bar{a}_{i2}L^2)^{-1}$ for $i = 1, \ldots, 4$ of the lag polynomial $\bar{H}(L)$. According to the estimates in Table 4, the implied estimates of the autoregressive coefficients are $\bar{a}_{11} = 1.357$, $\bar{a}_{21} = 1.04$, $\bar{a}_{31} = 1.013$, $\bar{a}_{41} = 0.865$, $\bar{a}_{12} = 0.0423$, $\bar{a}_{22} = 0.094$, $\bar{a}_{32} = 0.067$, and $\bar{a}_{42} = 0.93$, while the estimates of the autoregressive roots are $z_{11} = 1.3191$, $z_{31} = 1.2189$, $z_{41} = 1.4934$, $z_{12} = (1.1454, -1.1171)$, $z_{22} = 11.285$, $z_{32} = 20.2507$, and $z_{42} = -9.5227$.

The impulse response functions can be obtained from the representation in (3.2) only under the assumption that ξ_t is generated from the levels of some unobservable series y_t as $\xi_t = \bar{y}_t - \bar{y}_{t-1} + \bar{v}_t = \bar{H}(L)\bar{\varepsilon}_t + \bar{v}_t$. In this case, the unobservable levels under both (3.1) and (3.2) are determined subject to initial values for y_t (which depends on the initial condition for λ_{1t}) and for $\bar{y}_t$. With λ_{10} normalized as zero, the initial values of the elements of y_t are identical to their deterministic equilibrium values. But model (3.2) does not have sufficient information to determine the initial value of $\bar{y}_t$. Without such information, I assume that both y_0 and $\bar{y}_0$ are equal to the zero vector and calculate the response of each variable to a unit impulse, beginning from the arbitrarily determined level of zero.

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