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Rational Expectations Models and the Aliasing Phenomenon

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This paper shows how the cross-equation restrictions delivered by the hypothesis of rational expectations can serve to solve the aliasing identification problem. It is shown how the rational expectations restrictions uniquely identify the parameters of a continuous time model from statistics of discrete time models.

Thanks are due to Ian Bain, who calculated the numerical example.

1. Introduction

This paper studies the identification of continuous time models from discrete time observations in the context of linear versions of the class of rational expectations dynamic economic models that Lucas [10], Lucas and Prescott [12], and Lucas and Sargent [13] have advocated. The endeavor of building models of this class is motivated by the desire to retain the capability of analyzing an interesting class of policy interventions that is promised by "structural" models, while meeting the criticisms of most econometric policy evaluation methods that were made by Lucas [10]. As Lucas [10], Lucas and Sargent [13] and Hansen and Sargent [6, 7, 8] have pointed out in several related contexts, even linear rational expectations models typically are characterized by sets of highly nonlinear cross-equation identifying restrictions, which to a large extent replace the linear (usually exclusion, usually within-equation) restrictions used to identify most existing econometric models. Analysis of these cross-equation restrictions and tractable procedures for implementing them have only begun to be developed. 1/ The purpose of this paper is to indicate how these cross-equation restrictions are useful in estimating continuous time rational expectations models. In particular, we illustrate in a simple context how the nonlinear cross-equation restrictions implied by rational expectations can serve to solve the "aliasing" identification problem involved in deducing parameters of a continuous time model from data that are sampled at discrete points in time.

Sims [20] and Geweke [4] have discussed the aliasing problem in a general context that is relatively free of the use of any explicit economic theory. Sims's approach was to characterize how the projection of $y(t)$ on an entire process $\{x(s), s \varepsilon (-\infty, \infty), s \text{ real}\}$ in continuous time is related to the discrete time projection of $y(t)$ on $\{x(s), s \varepsilon (-\infty, \infty), s \text{ an integer}\}$. Sims supplied conditions under which the discrete time lag distribution "looks like" the underlying continuous time lag distribution.

P.C.B. Phillips [15, 16, 17] studied the aliasing problem in a context that was more restricted than Sims's both in terms of studying a narrower class of continuous time stochastic processes, and in terms of building models with more a priori theoretical restrictions. Phillips [17] showed how various kinds of linear restrictions of the Cowles Commission variety could be used to identify a continuous time model from discrete time sampled observations. Phillips's [17] work is the closest precedent of this paper. We analyze a class of linear rational expectations models that give rise to systems of stochastic differential and difference equations that closely resemble the forms of Phillips's systems. The big difference is that we shall analyze identifying restrictions of quite a different variety than those studied by Phillips.

Section 2 of this paper is devoted to deriving a convenient form for the "decision rule" or "rational expectations equilibrium" that solves a multivariable continuous time linear - quadratic stochastic optimum problem. This derivation is of interest in its own right, since it leads to compact expressions for the optimum decision rules that prove useful not only for theoretical analysis but for econometric implementation. Section 3 of the paper discusses the aliasing problem as formulated by Phillips [17], and corrects an assertion made by Phillips [17]. In the finite parameter model used both by Phillips and ourselves, the identification problem due to aliasing is not as bad as Phillips asserted: instead of there being a countable infinity of observationally equivalent continuous time models, there is at most a finite number. This modification of Phillips's results is of practical importance, since in finite parameter models one may sometimes encounter examples in which, even with no restrictions on the continuous time model, there is a unique continuous time model consistent with the data. Section 4 illustrates how the nonlinear rational expectations restrictions derived in section 2 can be used to overcome the aliasing identification problem. Our conclusions are stated in section 5.

This paper does not deal with questions of estimation. A sequel to this paper [8] deals with estimation issues in the convenient context of somewhat more general models of the continuous time stochastic processes. That greater generality is achieved at the cost of obscuring the identification issue. It is for this reason and also because we believe it helpful to the reader that our setup matches that of Phillips [17] as closely as possible, that we choose to concentrate on identification using the setup of this paper.

2. The Optimum Decision Rule

An agent maximizes over strategies for $k(t)$ the criterion

$$(1) \qquad E_0 \int_0^\infty F(k(t), \dot{k}(t), t, J(t), f(t))dt$$

where

$$(2) \qquad F(k(t), \dot{k}(t), t, J(t), f(t)) = \{[f_0 + f_1(t)]'k(t) - k(t)'Qk(t) - J'(t)\dot{k}(t) - \dot{k}'(t)H\dot{k}(t)\}e^{-rt}$$

Here $k(t)$ is an $(n\times 1)$ vector of stocks, $J(t)$ an $(n\times 1)$ vector of relative prices of stocks, $f_1(t)$ an $(n\times 1)$ vector of random shocks to productivity; while f_0 is an $(n\times 1)$ vector of positive constants; Q and H are symmetric positive definite matrices, and r is a fixed positive discount rate. The vectors $J(t)$ and $f_1(t)$ are the first n elements of the $(p\times 1)$ and $(q\times 1)$ vectors $z(t)$ and $y(t)$, respectively, where $p \geq n$, $q \geq n$. The vectors $z(t)$ and $y(t)$ are governed by the first order linear stochastic differential equations[2/]

$$(3) \qquad \dot{z}(t) = Cz(t) + \zeta(t)$$

$$(4) \qquad \dot{y}(t) = Gy(t) + \xi(t)$$

where $\zeta(t)$ and $\xi(t)$ are mutually uncorrelated continuous time white noise processes, with means of zero and covariances $E\zeta(t)\zeta(t-\tau)' = V_1\delta(t-\tau)$ and $E\xi(t)\xi(t-\tau)' = V_2\delta(t-\tau)$, where V_1 and V_2 are positive semi definite matrices.[3/] We assume that the eigenvalues of C and G do not exceed $r/2$ in real part. The maximization in (1) is over linear "nonanticipative" contingency plans which in "feedback form" can be expressed as

(5) $$\dot{k}(t) = L(k(t), z(t), y(t)) .$$

The maximization problem (1) can be viewed as a stochastic, linear-quadratic, multiple variable version of the Lucas [11], Treadway [21], Gould [5], Mortensen [14] costly adjustment model. Linear, multiple factor versions in continuous time of the Lucas-Prescott [12] equilibrium model of investment also lead to problems of this form, where the optimum problem (1) is solved by a fictitious "social planner" who uses it to compute a rational expectations competitive equilibrium.[4/] A univariate model similar to problem (1) was used by Geweke [20] to motivate some econometric interpretations.

To solve the problem, we invoke the widely known fact that the solution to (1) in feedback form (5) can be obtained simply by solving the certainty problem that emerges when we set V_1 and V_2 equal to zero in (3) and (4), so that $\zeta(t) = 0$, and $\xi(t) = 0$ identically in time.[5/] We find it convenient to solve (1) in the certain case by using the calculus of variations. The Euler equations are

(6) $$\frac{\partial F}{\partial k} = \frac{d}{dt}\frac{\partial F}{\partial \dot{k}}$$

Calculating the indicated derivatives in (2) and rearranging leads to the second order linear differential equation

(7) $$HD^2k(t) - rHDk(t) - Qk(t) = \frac{1}{2}[rJ(t)-f_1(t) - DJ(t)] ,$$

where D is the time derivative operator $D = \frac{d}{dt}$. For convenience, we have set $f_0 = 0$. Given the results to follow, the reader can easily modify the solution to handle the case where $f_o \neq 0$. Define the transformed variables $\tilde{k}(t) = e^{-\frac{r}{2}t}\, k(t)$. In terms of the transformed vector of variables $\tilde{k}(t)$ we have

$$k(t) = e^{\frac{r}{2}t}\tilde{k}(t)$$

(8) $$Dk(t) = e^{\frac{r}{2}t}D\tilde{k}(t) + \frac{r}{2}e^{\frac{r}{2}t}\tilde{k}(t)$$

$$D^2k(t) = e^{\frac{r}{2}t}D^2\tilde{k}(t) + re^{\frac{r}{2}t}D\tilde{k}(t) + \frac{r^2}{4}e^{\frac{r}{2}t}\tilde{k}(t) .$$

Substituting (8) into (7) gives the modified Euler equation in terms of the transformed variables,

(9) $$-HD^2\tilde{k}(t) + [\frac{r^2}{4}H + Q]\tilde{k}(t) = -\frac{1}{2}e^{-\frac{r}{2}t}[rJ(t) - f_1(t) - DJ(t)]$$

Consider the Laplace transform of the linear operator that operates on $\tilde{k}(t)$ on the left side of (9), namely

$$-s^2H + [\frac{r^2}{4}H + Q] = S(s)$$

Evaluating this at $s = i\omega$ gives

$$S(i\omega) = H\omega^2 + \frac{r^2}{4}H + Q .$$

Since both H and Q are positive definite symmetric matrices, it follows that $S(i\omega)$ is positive definite for all ω on the real line. Therefore, as an implication of the factorization theorem for spectral density matrices (see Rozanov [16]), there exists a factorization of $S(i\omega)$ of the form

(10) $$(H\omega^2 + \frac{r^2}{4}H + Q) = [\alpha-\beta(i\omega)]^T[\alpha+\beta(i\omega)]$$

where α and β are each $n\times n$ matrices, and the zeroes of $\det(\alpha-\beta s)$ lie in the right half plane while the zeroes of $\det(\alpha+\beta s)$ lie in the left half plane. The factorization is unique up to premultiplication of α and β by a common unitary matrix.

Using the factorization (10) we write the Euler equation (9) in the operator notation

$$(11) \qquad [\alpha-\beta D]^T[\alpha+\beta D]\tilde{k}(t) = \frac{1}{2} e^{-\frac{r}{2}t}[DJ(t) + f_1(t) - rJ(t)] \ .$$

Since the zeroes of $\det(\alpha+\beta s)$ are less than zero in real part, and those of $\det(\alpha-\beta s)^T$ are greater than zero in real part, the solution that satisfies the transversality condition is

$$[\alpha+\beta D]\tilde{k}(t) = \frac{1}{2}[\alpha^T-\beta^T D]^{-1} e^{-\frac{r}{2}t}[DJ(k) + f_1(t) - rJ(t)] \ .$$

Solving for $D\tilde{k}(t)$ gives

$$(12) \qquad D\tilde{k}(t) = -\beta^{-1}\alpha\tilde{k}(t) + \frac{1}{2}\{\alpha^T\beta-\beta^T\beta D\}^{-1} e^{-\frac{r}{2}t}[DJ(t)+f_1(t)-rJ(t)] \ .$$

Notice that $[\alpha^T\beta-\beta^T\beta s]^{-1} = \beta^{-1}[\alpha^T - \beta^T s]^{-1}$, so that the zeroes $[\alpha^T\beta - \beta^T\beta s]$ are greater than zero in real part. Assume that the zeroes of $\det[\alpha^T - \beta^T s]$ are all distinct, and define $M(s) = [\alpha^T\beta - \beta^T\beta s]$, so that 6/

$$M(s)^{-1} = [\alpha^T\beta - \beta^T\beta s]^{-1} = \frac{\text{adj } M(s)}{\det M(s)}$$

(13)

$$= \frac{\text{adj } M(s)}{s_0(s-s_1)\ldots(s-s_m)}$$

where $\det M(s) = s_0(s - s_1)\ldots(s - s_m)$ and where s_j, $j=1,\ldots,m$ are the m zeroes of $\det M(s)$, each of which exceeds zero in real part. Using matrix partial fractions, we express (12) as

$$(14) \qquad M(s)^{-1} = \sum_{j=1}^{m} \frac{N_j}{(s-s_j)}$$

where $$N_j = \frac{\text{Adj } M(s_j)}{s_0 \prod_{i\neq j} (s_i - s_j)}$$

Substituting (14) into (12) gives

$$\text{(15)}\qquad D\tilde{k}(t) = -\beta^{-1}\alpha\tilde{k}(t) + \frac{1}{2}\sum_{j=1}^{m}\frac{N_j}{(D-s_j)}e^{-\frac{r}{2}t}[DJ(t) + f_1(t) - rJ(t)]$$

It is easy to verify that for $s_j > 0$, $(s-s_j)^{-1}$ is the Laplace transform of the function

$$\begin{array}{ll} -e^{+s_j t} & t \leq 0 \\ 0 & t > 0 \end{array}$$

Therefore (15) becomes

$$\text{(16)}\qquad D\tilde{k}(t) = -\beta^{-1}\alpha\tilde{k}(t) - \frac{1}{2}\sum_{j=1}^{m}N_j\int_0^{\infty}e^{-s_j v}\, e^{-\frac{r}{2}(t+v)}[DJ(t+v) + f_1(t+v) - rJ(t+v)]dv$$

Let u_1 and u_2 be vectors such that $J(t) = u_1 z(t)$ and $f_1(t) = u_2 y(t)$. Note that the solutions of (3) and (4) are, respectively,

$$z(t+v) = e^{Cv}\, z(t) + \int_t^{t+v} e^{C(t+v-\tau)}\zeta(\tau)d\tau$$

$$y(t+v) = e^{Gv}\, y(t) + \int_t^{t+v} e^{G(t+v-\tau)}\xi(\tau)d\tau$$

Therefore we have that the linear least squares forecasts are

$$\text{(17)}\qquad \begin{aligned} E_t z(t+v) &= e^{Cv}z(t) \\ E_t \dot{z}(t+v) &= Ce^{Cv}z(t) \\ E_t y(t+v) &= e^{Gv}y(t)\ . \end{aligned}$$

Using (17) and $J(t) = u_1 z(t)$ and $f_1(k) = u_2 y(t)$, we have

$$(18) \qquad E_t f_1(t+v) = u_2 e^{Gv} y(t)$$

$$(19) \qquad E_t[DJ(t+v) - rJ(t+v)] = u_1[C-rI]e^{Cv} z(t).$$

The "certainty equivalence principle" applies and implies that the linear least squares forecasts (18) and (19) can be substituted for the actual future values in (16) in order to obtain the correct decision rule for the uncertain problem. Making these substitutions we obtain

$$D\tilde{k}(t) = -\beta^{-1}\alpha\tilde{k}(t) - \frac{1}{2}\sum_{j=1}^{m} N_j e^{-\frac{r}{2}t} \int_0^\infty e^{-(s_j+\frac{r}{2})v} u_1[C-rI]e^{Cv} z(t)dv$$

$$- \frac{1}{2}\sum_{j=1}^{m} N_j e^{-\frac{r}{2}t} \int_0^\infty e^{-(s_j+\frac{r}{2})v} u_2 e^{Gv} y(t)dv \quad .$$

Using $\int_0^\infty e^{-(s_j+\frac{r}{2})v} e^{Cv} dv = -[C-(s_j + \frac{r}{2})I]^{-1}$, we have

$$(20) \qquad D\tilde{k}(t) = -\beta^{-1}\alpha\tilde{k}(t) + \frac{1}{2} e^{-\frac{r}{2}t} \sum_{j=1}^{m} N_j u_1 [C-rI][C-(s_j + \frac{r}{2})I]^{-1} z(t)$$

$$+ \frac{r}{2} e^{-\frac{r}{2}t} \sum_{j=1}^{m} N_j u_2 [G-(s_j + \frac{r}{2})I]^{-1} y(t) \quad .$$

Now recall that $\tilde{k}(t) = e^{-\frac{r}{2}t} k(t)$ so that $Dk(t) = e^{+\frac{r}{2}t} D\tilde{k}(t) + \frac{r}{2} e^{+\frac{r}{2}t}\tilde{k}(t)$. Substituting these expressions into (2) gives the following decision rule in terms of the original stock variables,

$$(21) \qquad Dk(t) = -[\beta^{-1}\alpha - \frac{r}{2}I]k(t) + \frac{1}{2}\sum_{j=1}^{m} N_j u_1 [C-rI][C-(s_j + \frac{r}{2})I]^{-1} z(t)$$

$$+ \frac{1}{2}\sum_{j=1}^{m} N_j u_2 [G-(s_j + \frac{r}{2})I]^{-1} y(t) \quad .$$

For convenience we repeat the following equations here

(3) $\dot{z}(t) = Cz(t) + \zeta(t)$

(4) $\dot{y}(t) = Gy(t) + \xi(t)$

(14a) $$[\alpha^T\beta-\beta^T\beta s]^{-1} = \sum_{j=1}^{m} \frac{N_j}{(s-s_j)}$$

where

(14b) $$N_j = \frac{\text{adj}[\alpha^T\beta-\beta^T\beta s_j]}{s_0 \prod_{i\neq j} (s_i-s_j)}$$

Equations (21), (3), (4), and (14a-b) summarize the restrictions that the decision theory imposes on the joint $\{k, z, y\}$ stochastic process.

We shall adopt the assumption that continuous time observations on $y(t)$, $k(t)$ and $z(t)$ are available to the agent, while only discrete-time point in time sampled observations on $k(t)$ and $z(t)$ are available to the econometrician. On this interpretation the term in $y(t)$ in the decision rule (21) becomes the source of the error term in the equation to be fit by the econometrician.[7/] For the purposes of the following section, it proves convenient to modify the preceding decision problem in the following way. We set $f_1(t) \equiv y(t)$, and suppose that f_1 itself is the derivative of the white noise ξ. Under this assumption (21), becomes

(22) $$Dk(t) = -[\beta^{-1}\alpha - \frac{r}{2} I]k(t) + \frac{1}{2} \sum_{j=1}^{m} N_j u_1 [C-rI][C-(s_j + \frac{r}{2})I]^{-1} z(t)$$

$$- \frac{1}{2} \sum_{j=1}^{m} N_j \xi(t) \ . \ ^{8/}$$

We adopt this assumption about $f_1(t)$ in order to avoid complicated models of the error term which would obscure but not essentially alter the main message of this paper about the aliasing problem. 9/ Also, this assumption is the one that makes our setup as comparable as possible to that of Phillips [17]. In a sequel to this paper, we analyze estimation of models under more general assumptions about the serial correlation properties of the y process.

The major aim of this paper is to show how the cross-equation restrictions embodied in (21), (3), (4) and (14) serve uniquely to identify the parameters of the continuous time model from discrete time data. Before turning to this task, we pause in the next section to characterize the dimension of the aliasing phenomenon in the general class of finite parameter models that we are using. In section 4, we then take up the identification problem in the context of the model formed by (21), (3), (4), and (14).

3. The Aliasing Problem in a Finite Parameter Model Reconsidered

In this section we study the problem of identifying parameters of a continuous time model from discrete time data, which is widely referred to as the aliasing problem. We begin with a frequency domain characterization of the phenomenon. Let f denote the spectral density matrix function of a continuous time covariance stationary vector stochastic process x, and let F denote the spectral density matrix function of the corresponding discrete time process obtained by observing x at integer points in time. It is well known that f and F are linked by the folding formula

$$F(\omega) = \sum_{n=-\infty}^{+\infty} f(\omega + 2\pi n) .$$

The aliasing phenomenon is that many choices of continuous time spectral densities f give rise to the same folded spectral density F. Since F summarizes the population covariance properties of x sampled at the integers, this implies that there is problem in inferring the function f from discrete time data.

Without additional restrictions, frequency domain characterizations via spectral density matrices are, at least implicitly, infinite parameter time series models. P.C.B. Phillips [17] has studied the aliasing problem in the context of finite parameter time domain representations. More specifically, he assumed a continuous time first order vector Markov process of the form

$$Dx(t) = Ax(t) + \varepsilon(t) \tag{23}$$

where ε is a continuous time vector white noise. The corresponding discrete time process has a first order autoregressive representation

$$(24) \qquad x(t) = Bx(t-1) + \eta(t)$$

where

$$(25) \qquad B = \exp A$$

$$\eta(t) = {}_0\int^1 \exp(Av)\,\varepsilon(t-v)\,dv,$$

By the white noise nature of ε, it follows that η is a discrete time vector white noise disturbance when sampled at the integers. The contemporaneous covariance matrix of $\eta(t)$ is

$$(26) \qquad W = {}_0\int^1 \exp(Av)\,V\exp(A'v)\,dv\ ,$$

where $E\varepsilon(t)\varepsilon(t-\tau)' = V\delta(t-\tau)$. As noted by Phillips [17], the covariance properties of x sampled at the integers are completely characterized by (B, W). Our goal is to identify the covariance properties of the continuous time process, which are completely characterized by (A, V). The version of the aliasing phenomenon considered by Phillips [17] is simply the fact that given (B, W) one cannot in general solve uniquely for (A, V) using equations (25) and (26). We investigate this problem in detail in this section of the paper, and modify Phillips's [17] characterization of the aliasing problem.

To begin, we consider equation (25) and ask the question of whether the matrix equation

$$(27) \qquad \exp A^* = B = \exp A$$

implies that $A^* = A$. Without restrictions on the matrix A^*, the answer is in general no. If the matrix A has complex eigenvalues then there is a countable infinity of matrices A^* that satisfy (27).

To see this, assume that the eigenvalues of A are distinct and write the spectral decomposition of A,

$$(28) \qquad A = T \Lambda T^{-1}$$

where Λ is a diagonal matrix of eigenvalues of A and T is a matrix of eigenvectors of A. Without loss of generality, we are free to assume that the first $M - 2\rho$ diagonal elements are real and that the remainder occur in complex conjugate pairs as $\lambda_{M-2\rho+1},\ldots,\lambda_{M-\rho}, \lambda_{M-\rho+1} = \bar{\lambda}_{M-\rho+1},\ldots,\lambda_M = \bar{\lambda}_{M-\rho}$, where the bar denotes complex conjugation. We assume that the eigenvalues of A do not differ by multiples of $2\pi i$. Following Phillips [17] and Coddington and Levinson [1], if a matrix A^* satisfies (27) then

$$(29) \qquad A^* = A + 2\pi i T \begin{pmatrix} 0 & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & -P \end{pmatrix} T^{-1}$$

where P is a ρ by ρ diagonal matrix of integers. Any choice of integers for the diagonal elements of P will give rise to a solution of the matrix equation (27).

Phillips [17] asserted that the pair (A, V) is identifiable in (B, W) if and only if the matrix A is identifiable in B. He reasoned this by an implicit claim that given a matrix A^* of the form specified in (29) it is possible to find a V^* such that

$$(30) \qquad {}_0\!\int^1 \exp(A^*v)V^*\exp(A^{*\prime}v)\,dv = W = {}_0\!\int^1 \exp(Av)V\exp(A'v)\,dv.$$

In fact Phillips's [17] equation (4) illustrates how to compute V^* from A^* and W. A problem with this reasoning is that V^* need not be positive semidefinite and thus need not be a valid covariance matrix. This indicates the presence of extra identifying information about A in the discrete innovation covariance matrix W via equation (26). Indeed, we shall show that this observation requires modifying Phillips's characterization of the identification problem. Phillips asserted that if A has complex eigenvalues, then without additional restrictions, there is a countable infinity of pairs (A^*, V^*) that are observationally equivalent to (A, V) given discrete time data. In fact, the number of pairs (A^*, V^*) that are observationally equivalent to (A, V) is, except for singular cases, at most finite and in some cases is equal to one even if A has complex eigenvalues. We proceed to substantiate this claim.

Following Phillips [17], from equation (30) we can deduce an alternative relationship

$$(31) \qquad \exp(A^*)V^* \exp(A^{*\prime}) - V^* = A^*W + WA^{*\prime} \qquad \underline{10/}$$

where " ' " denotes transposition and conjugation. By construction A^* has the same eigenvectors as A and thus we can write

$$(32) \qquad A^* = T\,\Lambda^*\,T^{-1}$$

where Λ^* is a diagonal matrix of eigenvalues of A^*. Using expression (29), it follows that

$$\Lambda^* = \Lambda + 2\pi i \begin{pmatrix} 0 & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & -P \end{pmatrix}$$

Also we can write

$$(33) \qquad \exp(A^*) = T \exp(\Lambda^*)T^{-1} .$$

Substituting (32) and (33) into (31), we have that

(34) $$T\exp(\Lambda^*)T^{-1}V^*T^{-1'}\exp(\Lambda^{*'})T' - V^* = T\Lambda^*T^{-1}W + WT^{-1'}\Lambda^{*'}T'$$

where again " ' " denotes transpose and conjugation. Premultiplying and postmultiplying both sides of (34) by T^{-1} and $T^{-1'}$, respectively, we obtain

(35) $$\exp(\Lambda^*)T^{-1}V^*T^{-1'}\exp(\Lambda^{*'}) - T^{-1}V^*T^{-1'}$$
$$= \Lambda^*T^{-1}WT^{-1'} + T^{-1}WT^{-1'}\Lambda^{*'} \quad .$$

Let

$$R^* = T^{-1}V^*T^{-1'} \qquad R = T^{-1}VT^{-1'}$$

$$S = T^{-1}WT^{-1'}$$

and substitute into (35). The result is

(36) $$\exp(\Lambda^*)R^*\exp(\Lambda^{*'}) - R^* = \Lambda^*S + S\Lambda^{*'} \quad .$$

Now R^* is positive semidefinite if and only if V^* is. We can solve for R^* in terms of S and Λ^*. Let

$$R^* = [r^*_{j,k}]$$

$$S = [s_{j,k}]$$

$$\Lambda^* = \mathrm{diag}(\lambda^*_1, \ldots, \lambda^*_M)$$

Equation (36) informs us that

$$\exp(\lambda^*_j)r^*_{j,k}\exp(\bar{\lambda}^*_k) - r^*_{j,k} = \lambda^*_j s_{j,k} + s_{j,k}\bar{\lambda}^*_k$$

or

$$r^*_{j,k} = \frac{(\lambda^*_j + \bar{\lambda}^*_k)s_{j,k}}{\exp(\lambda^*_j + \bar{\lambda}^*_k) - 1}$$

Let

$$R = [r_{j,k}] = T^{-1}VT^{-1'}$$

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_M) \ .$$

Using the fact that λ_j and λ_j^* differ by a multiple of $2\pi i$, it follows that

$$1 - \exp(\lambda_j^* + \bar{\lambda}_k^*) = 1 - \exp(\lambda_j + \bar{\lambda}_k)$$

and that

$$\lambda_j^* + \bar{\lambda}_j^* = \lambda_j + \bar{\lambda}_j \ .$$

Therefore

$$r_{j,j}^* = r_{j,j} \ ,$$

i.e., the diagonal elements of R are the same as the diagonal elements of R^*. The off-diagonal elements of R and R^* differ, however. By construction R is positive semidefinite since V is positive semidefinite. The question is whether V^* or equivalently R^* need be positive semidefinite.

We first investigate this in the context of an example. Suppose $M = 2\rho = 2$. Thus P is a scaler, and

$$\lambda_1^* = \lambda_1 + 2\pi i P$$

$$\lambda_2^* = \bar{\lambda}_1 - 2\pi i P$$

$$r_{1,1}^* = r_{1,1}$$

$$r_{2,2}^* = r_{2,2}$$

$$r_{1,2}^* = \frac{(2\lambda_1 + 4\pi i P)s_{1,2}}{-1 + \exp(2\lambda_1)} = r_{1,2} + \frac{4\pi i P s_{1,2}}{-1 + \exp(2\lambda_1)}$$

$$r_{2,1}^* = \bar{r}_{1,2}^* \ .$$

Now the determinant of R^* is given by

$$\text{(37)} \qquad \det R^* = r^*_{1,1} r^*_{2,2} - r^*_{1,2} r^*_{2,1}$$

$$= r_{1,1} r_{2,2} - \left| r_{1,2} + \frac{4\pi i P s_{1,2}}{-1 + \exp(2\lambda_1)} \right|^2$$

It is clear from equation (37) that so long as $s_{1,2}$ is not zero, for a sufficiently large choice of integer P the determinant of R^* will be negative and hence R^* will fail to be positive semidefinite. Thus when $M = 2\rho = 2$, there is at most a finite number of R^*'s and hence V^*'s that are positive semidefinite, except for the special cases in which T^{-1} diagonalizes W.

Now consider the general case. It turns out that we can apply the same logic as above to the two by two matrices,

$$\text{(38)} \qquad \begin{pmatrix} r^*_{M-2\rho+j,\ M-2\rho+j} & r^*_{M-2\rho+j,\ M-\rho+j} \\ r^*_{M-\rho+j,\ M-2\rho+j} & r^*_{M-\rho+j,\ M-\rho+j} \end{pmatrix}$$

for $j = 1,\ldots,\rho$. In order that R^* be positive semidefinite, each of the two by two matrices given in (38) must be positive semidefinite. Let

$$P = \operatorname{diag}\,[p_1,\ldots,p_\rho] \ .$$

As was shown above, there is only a finite number of integer choices of p_j for each j that give rise to positive semidefinite matrices R^* and V^*, so long as there are no zeroes in the 2ρ by 2ρ submatrix in the bottom right-hand corner of S. Thus, except for some special cases, there is at most a finite number of pairs (A^*, V^*) that satisfy both equation (27) and equation (30), and for which V^* is positive semidefinite.

The nature of expression (37) suggests that examples will occur in which there will be at most a <u>small</u> finite number of pairs (A^*, V^*) that satisfy (27) and (30), with V^* positive semidefinite. Such examples require that $s_{1,2}$ be sufficiently large relative to $r_{1,2}$. The upshot of this situation is that for certain values of the system matrix A, the identification problem may be much less drastic than was suggested by Phillips's characterization.

However, although the dimension of the identification problem is less than was suggested by Phillips, it is still generally present for the finite parameter models considered by Phillips and ourselves. Despite the preceding modifications of Phillips's characterization, we are in general in need of prior information about A in order to identify it uniquely from discrete data. In the next section, we describe how the cross-equation rational expectations restrictions can help achieve identification.

4. Rational Expectations Restrictions

In the previous section we discussed the aliasing phenomenon in the context of first order stochastic differential equations driven by white noise. In particular it was shown that a sufficient condition for the pair (A, V) to be identified in (B, W) is that $\hat{A}$ be identified in B. Phillips [17] has described a set of restrictions of the Cowles commission exclusion variety that are sufficient uniquely to identify A from B. While interesting and useful in various contexts, the restrictions considered by Phillips are not of the nonlinear cross-equation variety characteristic of the class of rational expectations models exemplified by the model described in section 2. As it turns out, the cross-equation restrictions imposed on the continuous time model by rational expectations are sufficient to identify A from B under general conditions. We shall argue this in the context of the theoretical model derived in section 2. More precisely, we shall study the continuous time model of the joint $\{k, z\}$ process

$$(22) \qquad Dk(t) = -[\beta^{-1}\alpha - \frac{r}{2} I]k(t) + \frac{1}{2}\sum_{j=1}^{m} N_j u_1 [C-rI][C-(s_j + \frac{r}{2})I]^{-1} z(t) - \frac{1}{2}\sum_{j=1}^{m} N_j \xi(t).$$

$$(3) \qquad Dz(t) = Cz(t) + \zeta(t)$$

where ζ and ξ are continuous time white noises and the N_j's obey

(14b) $$N_j = \frac{\mathrm{adj}\,[\alpha^T\beta-\beta^T\beta s_j]}{s_0 \prod_{i\neq j} (s_i-s_j)}$$

where $\det[\alpha^T\beta-\beta^T\beta s] = s_0(s-s_1)\ldots(s-s_m)$. We can write (22) and (3) as the joint first-order linear stochastic differential equation

(24) $$Dx(t) = Ax(t) + \varepsilon(t)$$

where

(38) $$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

$$A_{11} = -\,[\beta^{-1}\alpha - \tfrac{r}{2}I]$$

$$A_{12} = \frac{1}{2}\sum_{j=1}^{m} N_j u_1[C-rI][C-(s_j + \tfrac{r}{2})I]^{-1}$$

$$A_{22} = C$$

$$x(t) = \begin{pmatrix} k(t) \\ z(t) \end{pmatrix}, \qquad \varepsilon(t) = \begin{pmatrix} -\frac{1}{2}\sum_{j=1}^{m} N_j\xi(t) \\ \zeta(t) \end{pmatrix}$$

As noted in section 3, the discrete time model that describes observations on $x(t)$ sampled at points in time separated by one unit of time is

(24) $$x(t) = Bx(t-1) + \eta(t)$$

where

(25) $$B = \exp A$$

$$\eta(t) = \int_0^1 e^{Av}\,\varepsilon(t-v)dv.$$

By virtue of the serial uncorrelatedness of η, it follows that with discrete time data the parameters in B can be estimated consistently by least squares or various least squares based methods, such as approximations to generalized least squares. Given the matrix B, is it possible uniquely to determine the parameters the matrix A? That is, does the matrix equation

$$\exp A^* = B = \exp A \tag{27}$$

imply that $A^* = A$. Following the argument in section 3, assume that the eigenvalues of A are distinct and write the spectral decomposition of A,

$$A = T \Lambda T^{-1} \tag{28}$$

where Λ is a diagonal matrix of eigenvalues of A and T is the matrix whose columns are eigenvectors of A. Partition the matrices T and Λ in the eigenvalue decomposition of A conformably with A so that

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}$$

It is readily verified that $A_{11} = T_{11}\Lambda_1 T_{11}^{-1}$ and $A_{22} = T_{22}\Lambda_2 T_{22}^{-1}$, so that Λ_1 and Λ_2 are the diagonal matrices of the eigenvalues of A_{11} and A_{22}, respectively. Now let the first $M_1 - 2\rho_1$ eigenvalues of Λ_1 be real, and the remainder occur in complex conjugate pairs as $\lambda_{1,\, M_1-2\rho_1+1}, \ldots, \lambda_{1,\, M_1-\rho_1}, \lambda_{1,\, M_1-\rho_1+1} = \bar{\lambda}_{1, M_1-2\rho_1+1}, \ldots, \lambda_{1, M_1} = \bar{\lambda}_{1, M_1-\rho_1}$, where the bar denotes complex conjugation and $0 \leq \rho_1 \leq [M_1/2]$ where

M_1 is the dimension of Λ_1. Further, let the first $M_2 - \rho_2$ eigenvalues of Λ_2 be real, and the remainder occur in complex conjugate pairs as $\lambda_{2,M_2-2\rho_2+1},\ldots,\lambda_{2,M_2-\rho_2}$, $\lambda_{2,M_2-\rho_{2+1}} = \bar{\lambda}_{2,M_2-2\rho_2+1},\ldots, \lambda_{2,M_2} = \bar{\lambda}_{2,M_2-\rho_2}$, where $0 \leq \rho_2 \leq [M_2/2]$ where M_2 is the demension of Λ_2. As in section 3 we assume that the eigenvalues of A do not differ by multiples of $2\pi i$. Then if a matrix A^* is to satisfy (27), it must be related to A by

$$(39) \qquad A^* = A + 2\pi i T \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -P_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -P_2 \end{pmatrix} T^{-1}$$

where P_1 and P_2 are any diagonal matrices whose diagonal elements are arbitrary integers, of dimension ρ_1 and ρ_2 respectively. In effect, (39) displays a class of perturbations of the complex eigenvalues of A which leave the relation $B = \exp A^*$ satisfied.

To show that the restrictions imposed on the model by rational expectations are sufficient to identify A from B under general conditions we shall use the special nature of the perturbations of A which are admissible under (39). In particular, notice that all A^*'s that satisfy (39) must have identical matrices of eigenvectors, that is T matrices, and can differ only in the imaginary parts of their complex eigenvalues. We shall indicate how the cross-equation restrictions imposed by rational expectations in effect make T_{12} a function of the eigenvalues Λ_1 and Λ_2, so that even the admissible perturbations of the eigenvalues of the kind permitted in (39) alter T_{12} and so lead to violation of (39). This will be enough to establish the existence of a unique inverse of $B = \exp A^*$.

Using the partitioned inverse formula

$$T^{-1} = \begin{pmatrix} T_{11}^{-1} & -T_{11}^{-1} T_{12} T_{22}^{-1} \\ 0 & T_{22}^{-1} \end{pmatrix}$$

we obtain the version of (28) appropriate for our problem

$$A = \begin{pmatrix} T_{11}\Lambda_1 T_{11}^{-1} & T_{12}\Lambda_2 T_{22}^{-1} - T_{11}\Lambda_1 T_{11}^{-1} T_{12} T_{22}^{-1} \\ 0 & T_{22}\Lambda_2 T_{22}^{-1} \end{pmatrix}$$

It follows that

(40) $$A_{12} = [T_{12}\Lambda_2 T_{22}^{-1} - A_{11}T_{12}T_{22}^{-1}] .$$

We use (38) and (40) to express the cross-equation restrictions implied by the model in the form

$$[T_{12}\Lambda_2 T_{22}^{-1} - A_{11}T_{12}T_{22}^{-1}] =$$

$$\frac{1}{2}\sum_{j=1}^{m} N_j u_1 [A_{22} - rI][A_{22} - (s_j + \frac{r}{2})I]^{-1}$$

or

(41) $$T_{12}\Lambda_2 - A_{11}T_{12} = \frac{1}{2}\sum_{j=1}^{m} N_j u_1 [A_{22} - rI][A_{22} - (s_j + \frac{r}{2})I]^{-1} T_{22} .$$

Our aim is to solve (41) for T_{12}. Let λ_{2j} be the i^{th} eigenvalue of Λ_2 and λ_{1k} the k^{th} eigenvalue of A_{11}. Then (41) is known to have a unique solution T_{12} if and only if

(42) $$\lambda_{2j} - \lambda_{1k} \neq 0$$

for <u>any</u> pair (j, k), such that $j \neq k$ (see Gantmacher [4]). From (38), $A_{11} = -[\beta^{-1}\alpha - \frac{r}{2} I]$. Recall that the zeroes of det $(\alpha + \beta s)$ are less than zero in real part. Notice that the zeroes of det $(sI + \beta^{-1}\alpha - \frac{r}{2} I)$ are less than $\frac{r}{2}$ in real part. Thus the eigenvalues of A_{11} are less than $\frac{r}{2}$ in real part. The eigenvalues of A_{22} have been <u>assumed</u> to be less than $\frac{r}{2}$ in real part. The upshot of these remarks is that condition (42) for the existence of a unique solution solution T_{12} of (41) is not necessarily met given the restrictions that we have on the eigenvalues of A_{11} and A_{22}. However, given the nature of A_{11} and A_{22}, failure of (42) to hold is a singular case which we shall assume does not obtain. Therefore we shall restrict ourselves to the case in which (41) uniquely determines T_{12}.

We proceed to exhibit an explicit representation of the solution for T_{12}. Write (41) as

(43) $$(-A_{11})T_{12} + T_{12}\Lambda_2 = \phi$$

where $$\phi = \frac{1}{2}\sum_{j=1}^{m} N_j u_j [A_{22} - rI][A_{22} - (s_j + \frac{r}{2})I]^{-1} T_{22} .$$

Equivalently we can express (43) as

$$(-A_{11} \otimes I)\text{vec } T_{12} + (I \otimes \Lambda_2) \text{ vec } T_{12} = \text{vec } \phi$$

where vec represents the vector formed by taking the direct sum of the rows of a matrix, and $\otimes$ denotes the Kronecker product. Given the restrictions on the eigenvalues of A_{11} and A_{22} it follows that the matrix

$$(-A_{11} \otimes I) + (I \otimes \Lambda_2)$$

is invertible and thus

$$\text{vec } T_{12} = [(-A_{11} \otimes I) + (I \otimes \Lambda_2)]^{-1} \text{ vec } \phi \quad . \tag{44}$$

Equation (44) can be used to investigate the question that we originally posed: will perturbations of the complex eigenvalues of A_{11} or A_{22} of the admissible class defined by (39) leave the T_{12} implicitly defined generally by (41) unaltered? The answer is in general no. Under the rational expectations restrictions, T_{12} is a function of the eigenvalues of A_{11} and A_{22} and is altered even by "admissible" perturbations. This is sufficient to guarantee that subject to the rational expectation restrictions, $B = \exp A$ has a unique inverse expressing the parameters A of the continuous time model.

It is useful to illustrate the situation with an example in which the number of stocks n is one so that A_{11} is a scalar. Where $n = 1$, we have $A_{11} = -[\beta^{-1}\alpha - \frac{r}{2}]$ and $[\alpha\beta - \beta^2 s] = -\frac{1}{\beta^2}[s - \frac{\alpha}{\beta}]^{-1}$ so that $s_1 = \frac{\alpha}{\beta}$ and $N_1 = \frac{-1}{\beta^2}$. Therefore with $n = 1$, the decision rule can be written

$$Dk(t) = -[\beta^{-1}\alpha - \frac{r}{2}]k(t) - \frac{1}{2\beta^2} u_1[A_{22} - (\frac{\alpha}{\beta} + \frac{r}{2})I]^{-1} z(t) - \frac{1}{2\beta^2}\xi(t) \quad . \tag{45}$$

We let $r = 0$, $\beta = 1$, $\alpha = 1$. We set A_{22} and then computed A_{12} from

$$A_{12} = -\frac{1}{2\beta^2} u_1[A_{22} - (\frac{\alpha}{\beta} + \frac{r}{2})I]^{-1}$$

We first chose A_{22} so that the resulting A matrix was

$$(46) \qquad A = \begin{bmatrix} -1 & .42857 & .14286 \\ 0 & -1.0 & 1.0 \\ 0 & -1.0 & -2.0 \end{bmatrix}$$

The eigenvector and eigenvalue matrices of A are

$$T = \begin{bmatrix} 1 + 0i & -.24744 - .28571i & -.24744 + .28571i \\ 0 + 0i & .86603 - .5i & .86603 + .5i \\ 0 + 0i & 0 + i & 0 - i \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -1 + 0i & 0 & 0 \\ 0 & -1.5 + .86603i & 0 \\ 0 & 0 & -1.5 - .86603i \end{bmatrix}$$

We then perturbed (46) with an "admissible perturbation" according to (39). The new A* matrix was

$$(47) \qquad A^* = \begin{bmatrix} -1.0 & .11554 & .14392 \\ 0 & 2.62760 & 8.25520 \\ 0 & -8.25520 & -5.62760 \end{bmatrix}$$

with eigenvectors and eigenvalues

$$T^* = \begin{bmatrix} 1 & .01102 - .01477i & .01102 + .01477i \\ 0 & .86603 - .5i & .86603 + .5i \\ 0 & 0 + i & 0 - i \end{bmatrix}$$

and

$$\Lambda^* = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1.5 + 7.14921i & 0 \\ 0 & 0 & -1.5 - 7.14921i \end{bmatrix}$$

The matrix B corresponding to (46) (i.e. B = exp A) is

$$(48) \qquad B = \begin{bmatrix} .36788 & .11988 & .08169 \\ 0 & .24269 & .19627 \\ 0 & -.19627 & .04642 \end{bmatrix}$$

while the $B^* = \exp A^*$ corresponding to (47) is

$$(49) \qquad B^* = \begin{bmatrix} .36788 & .00006 & .00520 \\ 0 & .24269 & .19627 \\ 0 & -.19627 & .04642 \end{bmatrix} .$$

Notice that while the B_{11}, B_{21} and B_{22}'s are identical in and (49), the B_{12}'s differ. This reflects the cross-equation restrictions at work achieving identification. Notice how the experiment perturbs the eigenvalues of A_{22} admissibly, i.e. by adding and subtracting $2\pi i$ to elements of the pair of eigenvalues in A. The parts of the eigenvectors in T_{11}, T_{21}, and T_{22} are left unchanged by this perturbation, but T_{12} is altered. It is the alteration of T_{12} under the perturbation that leads B_{12} to be altered.

4. Conclusions

This paper indicates how the cross-equation restrictions delivered by dynamic economic theory of the rational expectations variety can serve to identify a continuous time model from discrete time observations. In effect, the dynamic economic theory underlying the cross equation restrictions provides enough prior information to overcome the aliasing problem. These restrictions are highly nonlinear and characteristically apply across equations.

This paper is intended as a prologue to further work that developes econometrically practical methods for estimating continuous time rational expectations models from discrete time data. A sequel to this paper [8] is devoted to estimation issues. [11/]

Footnotes

1. See Hansen and Sargent [6, 7] for discussion of the econometrics of discrete-time rational expectations models.

2. For an introduction to linear stochastic differential equations, see Kwakernaak and Sivan [9].

3. Here δ is the Dirac delta generalized function. See Kwakernaak and Sivan [9] for an introduction to its properties and uses.

4. For some examples, see Hansen and Sargent [6, 7].

5. This is proved, for example, in Kwakernaak and Sivan [9].

6. Our treatment of the derivation of the decision rule, and in particular of the representation of $M(s)^{-1}$ by partial fractions parallels the treatment of the discrete time case to be found in Hansen and Sargent [7].

7. This model of the error term is proposed and analyzed for discrete time models by Hansen and Sargent [6].

8. The assumption f_1 is the derivative of the white noise ξ is admittedly contrived in order that the decision rule will have a white noise disturbance. To see that (22) is the correct expression for the decision rule, note that

$$\int_0^\infty e^{-(s_j + \frac{r}{2})v} D\xi(t+v)dv = \xi(t) + (s_j + \frac{r}{2}) \int_0^\infty e^{-(s_j + \frac{r}{2})v} \xi(t+v)dv.$$

Taking conditional expectations as of time period t we obtain

$$E_t \int_0^\infty e^{-(s_j + \frac{r}{2})v} D\xi(t+v)dv = \xi(t).$$

Given that f_1 is not plysically realizable, we are not proposing this as a plausible model of a disturbance term.

9. See Hansen and Sargent [6] for a treatment showing how to generalize things beyond the white noise assumption for the case of a discrete time model.

10. See Phillips [17] page 354. Equation (31) is readily deduced from results of Kwakernaak and Sivan [9, p. 100–103] as follows. Consider the stochastic differential equation system of the form

(23) $Dx(t) = Ax(t) + \varepsilon(t), \quad E\varepsilon(t)\varepsilon(t-s)' = V\delta(t-s)$

with initial condition $x(0) = 0$. Kwakernaak and Sivan [9, p. 103, eqn. 1–514] show that $x(t)$ has covariance matrix

(*) $Q(t) = \int_0^t e^{As} V e^{A's} ds$

where $Q(t) = Ex(t)x(t)'$. They also show (p. 101, eqn. 1–502) that $Q(t)$ obeys the differential equation

(**) $\dot{Q}(t) = AQ(t) + Q(t)A' + V.$

Evaluating $\dot{Q}(t)$ from (*) by using Leibniz's rule and equating the result with the right side of (**) yields, for $t = 1$, equation (31).

11, This sequel is based on the work of A. W. Phillips [18], Hansen and Sargent [6, 7], and the present paper.

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