Federal Reserve Bank of Minneapolis
Research Department Staff Report 268

This version: January 2001

On the Irrelevance of the Maturity Structure of Government Debt Without Commitment

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ABSTRACT

This paper presents a government debt game with the property that if the timing of debt auctions within a period is sufficiently unfettered, the set of equilibrium outcome paths of real economic variables given the government has access to a rich debt structure is identical to the set of equilibrium outcome paths given the government can issue only one-period debt.

*The author would like to thank Andrew Atkeson, Harold Cole, Patrick Kehoe, and Narayana Kocherlakota for many useful comments. The views expressed herein are the those of the author and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

1. Introduction

Several observers have argued that short-term debt played an important role in the recent financial crises in Mexico and East Asia or, more generally, that short-term debt allows the possibility of bad equilibrium outcomes which would not exist under longer maturity structures. In particular, Cole and Kehoe (2000), Calvo (1988), and Alesina, Prati, and Tabellini (1990) argue that, in a world without commitment, governments can face financial crises as an equilibrium outcome if they issue too much short-term debt and that these crises cannot occur if they do not issue short-term debt. The natural policy prescription for such a situation is that governments should simply avoid issuing short-term debt.

Here, I question the theoretical basis of these arguments against short-term debt. In the above papers, the inability of a government to commit is crucial since it is relatively straightforward to show that under an assumption of full commitment by governments toward future policy, any allocation which can be achieved by a complicated portfolio of government bonds can be achieved by a government which issues only short-term contingent debt. This paper presents a result which parallels the known result under full commitment, but this time for a world without commitment: Any allocation which can be achieved by a complicated portfolio of government bonds can be achieved by a government which issues only short-term contingent debt.

Formally, I get this equivalence result by considering the set of equilibria of two games. In the first, a government can make payments to households conditional on their holdings of government-issued paper printed in the preceding period. In the second, the government can make payments to households conditional on their holdings of government-issued paper printed at all previous dates. I find that in terms of real outcomes, the set of equilibria for the two games is identical.

There are two key differences between my work and others' in this area (such as Cole and Kehoe [2000]). First, I assume a general timing structure for debt auctions and debt repayments. This allows one to design the timing of debt auctions and debt repayments in each environment to mimic the incentives a government faces when in the other. Second, I assume that default is a purely expectational phenomenon. That is, if a government defaults on its debt, while lenders may update their beliefs regarding future repayments, the physical world does not change.

I present the equivalence result as two theorems: (1) Any real outcome of a game with only one-period debt can be achieved as an equilibrium of a game with many-period debt, and (2) vice versa: any real outcome of a game with many-period debt can be achieved as an equilibrium of a game where only one-period debt is available.

The logic of the first result is simple and robust: To replicate an equilibrium (good or bad) from a world with only one-period debt in a world where many-period debt is available, one simply gives lenders an expectation that longer-period debt will not be repaid. Given this expectation, a government has no incentive to deviate and issue long-term debt.

The second result is more difficult. When short-term debt is used to replicate the net flows and household incentives of a debt scheme involving long-term debt, large debt rollovers may be required. This can affect the incentives of the government regarding default. If these large debt rollovers are required to be done in one step, then the size of the required debt auction may be too large for the government to resist the temptation to default. However, if the timing structure is generalized to allow a finite length sequence of smaller debt auctions, then the government can be made to never have more real resources on hand than it does under the long-term debt equilibrium. This allows the replication of any long-term debt equilibrium as an equilibrium in a world with only short-term debt. These two results then allow a simple characterization, independent of the term structure of debt, of when a debt scheme can be implemented as an equilibrium.

2. The Economy

Consider a world similar to those in Lucas and Stokey (1983) and Chari and Kehoe (1993), where an infinitely lived government faces a continuum of infinitely lived households with unit mass. In each period $t = 1, \ldots, \infty$, households care about their own leisure, their private consumption of the single consumption good, and the level of government consumption. A household's per-period utility over certain outcomes is given by

$$u(c_t) - w(\ell_t) + v(G_t, \alpha_t),$$

where c_t and ℓ_t represent private consumption and labor, G_t represents the common level of government services, and α_t represents a common shock to preferences for government services. The shock α_t is assumed to be observable, take on a finite number of values, and follow a one-stage Markov process represented by $\pi(\alpha'|\alpha)$ with α_1 given. Assume that $u : \mathbf{R}_+ \to \mathbf{R}$ is increasing, strictly concave, twice differentiable and unbounded below. Labor ℓ is assumed to be bounded above by unity and below by zero with $\lim_{\ell \to 1} w(\ell) = -\infty$ and $\lim_{\ell \to 0} w'(\ell) = 0$. Finally, assume $\lim_{G \to 0} v(G, \alpha) = -\infty$ for all α. Over a lifetime, households care about the expected discounted value of the stream of per-period utilities with the discount parameter denoted by $\beta < 1$.

The government is assumed to be benevolent. It discounts the future at the same rate as households and its per-period utility is given by

$$u(C_t) - w(L_t) + v(G_t, \alpha_t),$$

where C_t and L_t represent the mean levels of consumption and labor. The assumption of benevolence is purely for notational convenience. All results go through for other utility functions for the government.

Output is assumed to be linear in labor—one unit of labor produces one unit of the consumption good, which can be used for either private or government consumption in the period it was produced. No storage (productive or otherwise) is available.

The government is assumed to be able to make lump-sum transfers to households, but cannot lump sum tax. To fund government consumption or lump-sum transfers, the government can linearly tax labor (up to an upper bound $\overline{\tau} < 1$) or "borrow" from households. More specifically, the government has the ability to print pieces of paper (bonds) distinguishable by the date they were printed. (The issue date is written on the bond). The government can (in a manner to be made more explicit later) auction this paper to households for the consumption good and can make nonnegative transfers of the consumption good to households proportional to their holdings of paper printed at previous dates. The government can, of course, write anything else on this paper that it wishes to (such as at what dates and under what circumstances it "promises" to make a transfer of the consumption good conditional on household holdings of the paper), but since the government cannot commit to future actions, such writing is irrelevant. In the next sections, two specific game structures are considered regarding the auctioning and repayment of this debt.

3. One-Period Debt

First, consider a simple game where the government is restricted to issue only one-period debt. Formally, paper is assumed to last from the beginning of period t until the end of period $t+1$, when it dissolves. (At any point in time there are only two distinguishable types of paper: that printed last period and that printed this period). Time is divided into periods $t = 1, \dots, \infty$, and the periods themselves are divided into subperiods $i = 1, \dots, I$. These subperiods correspond to the timing of auctions and repayments, and not to production or consumption, and thus can be considered as taking arbitrarily little real time.

At the beginning of each period t, households hold nonnegative amounts of government paper printed in period $t-1$. The aggregate amount of this is assumed, without loss, to be unity. (In period $t = 1$, each household owns a unit of paper dated $t = 0$.) In each period, the government moves first by announcing a tax rate on labor, with households subsequently choosing labor $\ell_t \geq 0$ with an average value of L_t.[1] The collection of labor tax revenue is assumed to occur simultaneously with production. Thus, at this point, the government has $\tau_t L_t$ units of the consumption good, and a household which chose ℓ_t has $(1-\tau_t)\ell_t$ units of the consumption good.

Next the government auctions fraction $q_{t,1}$ of period t paper to households. This results in a transfer from households to the government of $q_{t,1}P_{t,1}$ units of the consumption good, where $P_{t,1}$ denotes the price of a unit of period t paper at the first auction ($i = 1$). The government then chooses a lump-sum payment to households $X_{t,1} \geq 0$ and a payment to

[1] The timing ensures that labor taxes are distortionary. Given that preferences for government spending change over time, this introduces a motive to use debt in order to smooth labor taxes.

holders of period $t-1$ paper (per unit of paper) $R_{t,1} \geq 0$. To rule out Ponzi type schemes, there exists some large number $\overline{R}$ such that $R_{t,1} \leq \overline{R}$ for all t. This cycle of auctioning off new paper and the government making lump-sum payments and payments to holders of old paper is then repeated $I-1$ times.

Given this structure, government consumption is

$$G_t = \tau_t L_t + \sum_{i=1}^{I} q_{t,i} P_{t,i} - \sum_{i=1}^{I} X_{t,i} - \sum_{i=1}^{I} R_{t,i}$$

and household consumption is

$$C_t = (1-\tau_t) L_t - \sum_{i=1}^{I} q_{t,i} P_{t,i} + \sum_{i=1}^{I} X_{t,i} + \sum_{i=1}^{I} R_{t,i}.$$

After consumption occurs, the shock to preferences for the next period α_{t+1} is realized along with a publicly observed realization of a sunspot variable γ_{t+1} with a uniform $[0,1]$ distribution. The payoff to the government this period is then $u(C_t) - w(L_t) + v(G_t, \alpha_t)$.

I will consider only symmetric equilibria where all households take the same actions along the equilibrium path. Let an exogenous history $\theta_t = (\alpha_t, \gamma_t)$ with $\theta^t = \{\theta_1, \ldots, \theta_t\}$. Let a complete history (including government actions) be denoted $h_t = \{\tau_t, \{X_{t,i}, R_{t,i}\}_{i=1}^{I}, \theta_{t+1}\}$ with $h^t = \{h_0, \ldots, h_t\}$. A strategy for this game is a specification of prices $P_{t,i}$, government actions $(\tau_t, X_{t,i}, R_{t,i})$, and household actions L_t (consumption is residual), each as functions of the observed history of exogenous shocks θ^t and government actions. A symmetric sequential equilibrium (or SSE) is such a specification of these history-dependent prices, household, and government actions such that i) after all histories h^t, no household can improve its lifetime payoff by deviating when taking prices, the actions of the other households and the government as specified, and ii) after all histories h^t, the government cannot improve its lifetime payoff taking prices and the actions of households as specified.

Several characteristics of an SSE can be immediately deduced. First, households are optimizing only if along the path of play

$$P_{t,i} = P_{t,j} \tag{1}$$

(so $P_t \equiv P_{t,i}$). Second, it is necessary that for all histories (on and off the path of play), the household's marginal condition on savings is satisfied, or,

$$u'(C_t)P_t = \beta \ E_{\theta_{t+1}}[u'(C_{t+1}) \sum_{i=1}^{I} R_{t+1,i}|\theta_t]. \tag{2}$$

Finally, it is necessary that the household's marginal condition on labor versus consumption is satisfied for all histories, or,

$$u'(C_t)(1 - \tau_t) = w'(L_t). \tag{3}$$

One can immediately characterize the worst equilibrium of this game from the perspective of the government. Suppose $R_{t,i} = 0$ and $P_{t,i} = 0$ at every history of the game (including past deviations by the government), $q_{t,1} = 1$, and $X_{t,i} = 0$ for all $i \in \{1, \ldots, I\}$. Given this, all links between periods are severed and one can deliver values for τ_t and L_t by solving the static game given α_t. Path play of this static game is the solution to

$$\max_{\tau,L} u((1-\tau)L) - w(L) + v(\tau L, \alpha_t) \tag{4}$$

subject to L solving

$$u'((1-\tau)L)(1-\tau) = w'(L). \tag{5}$$

Define $u^A(\alpha)$ to be the maximized value of (4) subject to (5) and $\underline{V}(\alpha)$ such that

$$\underline{V}(\alpha) = u^A(\alpha) + \sum_{\alpha'} \pi(\alpha'|\alpha)\underline{V}(\alpha').$$

LEMMA 1. A balanced budget rule ($R_{t,i} = 0$, $P_{t,i} = 0$ and $X_{t,i} = 0$, $i \in \{1, \ldots, I\}$, at every history of the game, including past deviations by the government, with (τ_t, L_t) solving (5) given α_t) is an equilibrium. Further, all other equilibria are weakly better (for the government) after all histories.

Proof. If $R_{t,i} = 0$ for all histories, household optimization implies $P_{t,i} = 0$. Further, if all households hold an equal share of the government debt, $R_{t,i}$ acts exactly as the lump sum rebate. Since labor taxes are distortionary, standard arguments imply setting $R_{t,i} = 0$ is optimal.

Next, suppose there exists an equilibrium with a lifetime payoff to the government worse than $\underline{V}(\alpha_t)$. Given the concavity assumptions, $L(\alpha)$ is the unique best response to $\tau(\alpha)$ (where $(L,\tau)(\alpha)$ is defined by (4)). Thus, the government can unilaterally achieve $\underline{V}(\alpha_t)$, and thus if it deviates to the balanced budget rule, it receives a higher continuation payoff. ■

Given Lemma 1, the government is optimizing when setting taxes iff for all histories,

$$E[\sum_{s=0}^{\infty}\beta^s(u(C_{t+s})-w(L_{t+s})+v(G_{t+s},\alpha_t))]\geq \underline{V}(\alpha_t). \tag{6}$$

The condition on government optimization regarding debt payments is as follows. Let $Y_{t,i}$ be the amount of resources the government has on hand after the ith auction (but before the ith payment), or $Y_{t,i}=\tau_t L_t+\sum_{j=1}^{i}q_{t,j}P_t-\sum_{j=1}^{i-1}(X_{t,j}+R_{t,j})$. The government is optimizing when making payments $(X_{t,i},R_{t,i})$ iff for all histories and all $\hat{X}\geq 0$,

$$\begin{aligned} & E[\sum_{s=0}^{\infty}\beta^s(u(C_{t+s})-w(L_{t+s})+v(G_{t+s},\alpha_t))] \\ \geq\ & u(L_t-Y_{t,i}+\hat{X})-w(L_t)+v(Y_{t,i}-\hat{X},\alpha_t)\ +\ \beta\sum_{\alpha_{t+1}}\pi(\alpha_{t+1}|\alpha_t)\underline{V}(\alpha_{t+1}). \end{aligned} \tag{7}$$

That is, the government can receive the path payoff (the left hand side of (7)) or can default, set government spending weakly less than $Y_{t,i}$, and then receive the payoff associated with the balanced budget equilibrium.

The next lemma is useful for establishing bounds on prices.

Lemma 2. There exist $\underline{C}>0$ and $\underline{G}>0$ such that in all equilibria, $C_t\geq\underline{C}$ and $G_t\geq\underline{G}$ for all t and all histories.

Proof. A household can always unilaterally achieve the utility

$$\max_L \frac{1}{1-\beta}\Big(u((1-\overline{\tau})L)-w(L)\Big)$$

for the parts of its utility that it can affect. (It cannot affect government). That u is unbounded below then delivers $\underline{C}$. Likewise, the government can unilaterally achieve $\underline{V}(\alpha)$. That v is unbounded below then delivers $\underline{G}$. ■

That consumption is bounded below by $\underline{C} > 0$ implies that the marginal utility of consumption is bounded above by $u'(\underline{C})$. The marginal utility of consumption in any equilibrium is bounded below by $u'(1)$. Since the most owning an equal share of the paper in the economy could deliver is a payment of $\overline{R}$ at all future dates, the price of a unit of paper cannot exceed $\overline{P} \equiv \frac{\beta}{1-\beta}\overline{R}\frac{u'(\underline{C})}{u'(1)}$.

4. Many-Period Debt

Consider now a more general (and thus complicated) game. At the beginning of each period $t \geq 1$, households hold nonnegative amounts of government paper printed in *all* previous periods. Paper is distinguishable only by the period t in which is was printed. The aggregate amount of paper printed in each period is assumed to be unity without loss of generality. (At $t = 1$, households are again assumed to each hold a unit of paper dated $t = 0$.) The government moves first by announcing the tax rate on labor τ_t, and subsequently, households choose labor (and thus production) $\ell_t \geq 0$ with an average value of L_t. As before, the collection of labor tax revenue is assumed to occur simultaneously with production.

Next, assume a market for the retrading of old paper with P_t^s denoting the price at t of paper printed in period s. In any symmetric equilibrium, there is no trade in these markets, but the period t price of paper printed in period s is established.

Once production and retrading have occurred, the government next auctions fraction $q_{t,1}$ of period t paper to households at price $P_{t,1}^t$. As before, the government next chooses a lump-sum payment to households $X_{t,1} \geq 0$ as well as a *vector* of payments $\{R_{t,1}^s\}_{s=0}^{t-1}$ to holders of paper printed in previous periods. Here $R^s_{t,1}$ is the amount paid, per unit of paper, to a holder of period s paper. For convenience, it is assumed that these payments are paid conditional on a household's holdings at the beginning of period t before retrading occurs. This cycle of auctioning off new paper and the government making lump-sum payments and payments to holders of old paper is then repeated $J-1$ times.

Given this structure, government consumption is

$$G_t = \tau_t L_t + \sum_{i=1}^{J} q_{t,i} P^t_t - \sum_{i=1}^{J} X_{t,i} - \sum_{s=0}^{t-1} \sum_{i=1}^{J} R^s_{t,i}, \tag{8}$$

and average (or symmetric) private consumption is

$$C_t = (1-\tau_t) L_t - \sum_{i=1}^{J} q_{t,i} P^t_t + \sum_{i=1}^{J} X_{t,i} + \sum_{s=0}^{t-1} \sum_{i=1}^{J} R^s_{t,i}. \tag{9}$$

After consumption occurs, the shock to preferences for the next period α_{t+1} is realized along with a publicly observed realization of a sunspot variable γ_{t+1} with a uniform $[0,1]$ distribution. The payoff to the government this period is then $u(C_t) - w(L_t) + v(G_t, \alpha_t)$.

This game has a definition of an equilibrium equivalent to that in the previous game and the same worst equilibrium. Now for each piece of paper, old and new, households must be willing to hold it. Thus for all histories to period t and $s \leq t$:

$$u'(C_t) P^s_t = \beta \; E_{\theta_{t+1}}[u'(C_{t+1})(P^s_{t+1} + \sum_{j=1}^{J} R^s_{t+1,j})|\alpha_t]. \tag{10}$$

The optimality conditions for government when setting taxes and making payments (equations (6) and (7)) are identical, except that amount of revenues the government has on hand after auction i, denoted $Y_{t,i}$ is now

$$Y_{t,i} = \tau_t L_t + \sum_{j=1}^{i} q_{t,j} P_t^t - \sum_{j=1}^{i-1} (X_{t,j} + \sum_{s=0}^{t-1} R_{t,j}^s).$$

The main difference between the two games is that with paper that last forever (as opposed to one period) it is possible to have, as an equilibrium, what looks like long term debt. For instance, a 30-period zero coupon bond would be an equilibrium expectation at date $t - 30$ that the government would make a payment at date t of some amount R_t^{t-30}. Such an expectation is impossible in the game where paper dissolves after one period.

5. Multiple Equilibria

It should be noted that both the one-period and many-period debt models allow for a continuum of equilibria. For instance, in both games, any specification of on-path consumption, labor, prices, and payments which satisfies the household first order conditions (with consumption between $\underline{C}$ and unity) can be supported as an equilibrium if conditions (6) and (7) hold by specifying reversion to the balanced budget equilibrium if the government ever deviates. Thus, if the rate of discount is not too high, the government can use debt to smooth labor taxes across shocks to α_t, as it would if it could commit to a conditional sequence of taxes and debt payments. On the other hand, there also exist sunspot equilibria as well where debt payments R_t depend on the realization of γ_t. Households will be willing to buy the government paper as long as the average return for holding government paper causes the appropriate intertemporal first order condition to hold, and the government will be willing to pay the high debt payments when called on to do so as long as this is preferred to reversion to the balanced budget equilibrium. Thus it is possible to construct equilibria in which a crisis (the realization of a particular range of sunspot values) causes the price of government paper to plummet. A key question is what difference the existence of multi-period debt has on whether the smoothing equilibria or the sunspot equilbria exist. The next section argues it makes no difference at all.

6. Constructing an Equivalence Between the Games

In this section the main theorems of the paper are stated and proved. Theorem 1 states that the outcome (in terms of real variables) of any equilibrium of the one-period debt game can be replicated as the outcome of the many-period debt game. Theorem 2 states the converse. Theorem 3 derives necessary and sufficient conditions on total gross flows and real variables (taxes, labor, and consumption), which determine whether there exists an incentive compatible timing structure for debt auctions.

Theorem 1. *If the number of subperiods of the many-period debt game J weakly exceeds the number of subperiods for the one-period debt game I, for each equilibrium of the one-period debt game, there is an equilibrium of the many-period debt game where path play for consumption, labor, taxes, and government consumption is identical.*

Proof. Let $\hat{X}$, $\hat{R}$, and $\hat{\tau}$ denote the government's path play; $\hat{P}_t$ denote path prices; and $\hat{L}_t$ denote household path play in the one-period debt game. This strategy uniquely defines a strategy in the many-period debt game as follows: on the path of play, the government in period t sets $R^s_{t,i} = 0$ for all $s < t-1$, $R^{t-1}_{t,i} = \hat{R}_{t,i}$, $X_{t,i} = \hat{X}_{t,i}$, and taxes the same. Next, likewise set $P^s_t = 0$ for all $s < t$ and $P^t_{t,i} = \hat{P}_t$ for all $i \in \{0, \ldots, J\}$. Finally, set L_t the same (again on the path of play). By construction, this gives the same payoff to the government as in the one-period game. Off path, let prices and actions correspond to the balanced budget rule equilibrium, which is weakly worse than any off-path payoffs in the one-period game equilibrium. Relative to the one-period debt game, the many-period debt game does give the government an expanded variety of possible deviations from this strategy and thus potentially higher one-shot deviation payoffs. (That is, when checking whether a deviation is profitable, one must now check to see if the government wishes to make R payments to holders of paper older than $t-1$.) Since, in a symmetric equilibrium, such a deviation acts as an extra lump-sum payment to households and in the one-period game the government moves last, the gain from the deviation can be achieved in the one-period debt game. ■

Note that this result holds for all $J \geq I \geq 1$ and thus obviously for $J = I = 1$. It does not depend on the ability of the government to hold more than one debt auction per period. The next result does.

THEOREM 2. *There exists an $I < \infty$ (where I is the number of sub-periods in the one-period debt game) such that for each equilibrium of the many-period debt game, there is an equilibrium of the one-period debt game where path play for consumption, labor, taxes, and government consumption is identical.*

Proof. Let $\{L_t(\theta^t), C_t(\theta^t)\}_{t=1}^{\infty}$, $\{\{P_t^s(\theta^t)\}_{s=0}^{t}\}_{t=1}^{\infty}$, $\{\tau_t(\theta^t)\}_{t=1}^{\infty}$, $\{\{X_{t,j}(\theta^t)\}_{i=1}^{J}\}_{t=1}^{\infty}$, and $\{\{\{R_{t,j}^s(\theta^t)\}_{s=0}^{t-1}\}_{j=1}^{J}\}_{t=1}^{\infty}$ be the stochastic outcome path of an equilibrium of the many-period game. Use these to construct an equilibrium of the one-period game. First, define play after a deviation by the government to be the balanced budget strategy. Next, construct path play from every history of shocks θ^t. For taxes, simply have the government set $\hat{\tau}_t$ (the taxes it imposes in the one-period debt game) equal to the level it imposes in the many-period debt game. Next, let the price of period t paper (which must be the same at all subperiods where the government auctions a positive amount of paper) be set by

$$\hat{P}_t(\theta^t) = \sum_{s=0}^{t} P_t^s(\theta^t).$$

This is the total price in period t (in the many-period debt game) of all the old and new paper in the economy. Next, require

$$\sum_{i=1}^{I} \hat{X}_{t,i}(\theta^t) = \sum_{j=1}^{J} X_{t,j}(\theta^t)$$

(or total lump-sum payments in the one-period debt game equal total lump-sum payments in the many-period debt equilibrium) and

$$\sum_{i=1}^{I} \hat{R}_{t,i}(\theta^t) = \sum_{s=0}^{t-1} P_t^s(\theta^t) + \sum_{s=0}^{t-1} \sum_{j=1}^{J} R_{t,j}^s(\theta^t),$$

or total bond payments in period t in the one-period debt game equal the period t value of all old paper in the many-period debt equilibrium plus the total bond payments in period t from the government to households.

Without specifying the timing of auctions and repayments, constructing prices and total repayments in this manner is sufficient for ensuring that household incentives are unchanged, and thus the requirement in the one-period debt game that households are optimizing is satisfied. The argument is as follows: Since taxes and lump-sum transfers (in total) are unchanged, the consumer's budget set is unchanged. (Gross transfers are changed since the current value of old paper is added to both the transfer from households to government and the transfers from government to households). Next, since taxes are the same, the current period tradeoff between consumption and labor is unchanged. This leaves the intertemporal incentives. Since consumer optimization is satisfied in the many-period debt equilibrium, for all histories and all $s \leq t$,

$$u'(C_t)P_t^s = \beta\ E[u'(C_{t+1})(P_{t+1}^s + \sum_{j=1}^{J} R_{t+1,j}^s)|\alpha_t]. \tag{11}$$

Summing this over all old paper and new paper in period t implies

$$u'(C_t)\sum_{s=0}^{t} P_t^s = \beta\ E[u'(C_{t+1})(\sum_{s=0}^{t}(P_{t+1}^s + \sum_{j=1}^{J} R_{t+1,j}^s))|\alpha_t]. \tag{12}$$

The intertemporal condition for household optimization in the one-period debt game is

$$u'(C_t)\hat{P}_t = \beta\ E_{\theta_{t+1}}[u'(C_{t+1})\sum_{i=1}^{I} \hat{R}_{t+1,i}|\alpha_t]. \tag{13}$$

This is then implied by the constructed values for $\hat{P}_t$ and $\sum_{i=1}^{I} \hat{R}_{t+1,i}$.

Like household incentives, government incentives when setting taxes are also unaffected by the timing of auctions and repayments. By construction, following path play yields the same payoff to the government in both games and a weakly lower deviation payoff in the one-period debt game, since no continuation equilibrium can yield a worse payoff than the balanced budget strategy. Since path play at this point in the game (choosing taxes) is weakly optimal in the many-period debt game, it is weakly optimal in the one-period debt game.

To check government incentives when making payments, one needs to specify the timing of auctions and repayments. Let

$$Z_t(\theta^t) = \max\{\tau_t(\theta^t)L_t(\theta), L_t(\theta^t) - C_t(\theta^t)\} > \underline{G}$$

and define $N(\theta^t)$ as the smallest integer weakly greater than $\hat{P}_t(\theta^t)/Z_t(\theta^t)$. Since Z_t is bounded below by $\underline{G}$ and P_t is bounded above by $\overline{P}$, N_t has a uniform upper bound. Next, set $\hat{q}_{t,1}(\theta^t) = 0$ and for $i = 2, \ldots, N(\theta^t) + 1$, $\hat{q}_{t,i}(\theta^t) = 1/N(\theta^t)$.

Define $\hat{X}_t(\theta^t) = \sum_{i=1}^{I} \hat{X}_{t,i}(\theta^t)$ and $\hat{R}_t(\theta^t) = \sum_{i=1}^{I} \hat{R}_{t,i}(\theta^t)$, and set $\hat{X}_{t,i}(\theta^t)$ and $\hat{R}_{t,i}(\theta^t)$ as follows. After tax revenues are collected, use these revenues to pay off as much of the "obligations" $\hat{X}_t(\theta^t)$ and $\hat{R}_t(\theta^t)$ as possible, but without exceeding these totals. (The division is irrelevant). Then, after each debt auction $i \geq 2$, continue to use the totality of these auction revenues to make these payments until the totals are correct.

To see that the government will not wish to deviate, let $Y_{t,i}(\theta^t)$ denote the amount of the consumption good the government has on hand after debt auction $i = 1, \ldots, I$, or

$$Y_{t,i}(\theta^t) := \tau_t(\theta^t)\ell_t(\theta^t) + \sum_{j=1}^{i} \hat{q}_{t,j}(\theta^t)\hat{P}_t(\theta^t) - \sum_{j=1}^{i-1}(\hat{X}_{t,j}(\theta^t) + \hat{R}_{t,j}(\theta^t)).$$

Since after a debt auction $\tau_t(\theta^t)$ and $\ell_t(\theta_t)$ are given, a maximal deviation yields a value of

$$\max_{X \geq 0} u^g(\ell_t(\theta^t) - Y_{t,i}(\theta^t) + X) - w^g(\ell_t) + v^g(Y_{t,i}(\theta^t) - X, \alpha_t) + \delta \sum_{\alpha'} \pi(\alpha'|\alpha)\underline{V}(\alpha').$$

This is monotonic in $Y_{t,i}$. Any period t allocation which can be achieved given a deviation with Y_1 on hand can be achieved at $Y_2 \geq Y_1$. (The government always has the ability to set government spending lower than $Y_{t,i}$ by making a lump-sum payment.)

Finally, the government can be trusted not to deviate when it has on hand $Z_t(\theta^t)$ since, under the multiperiod debt equilibrium, the government must have on hand at some point the maximum of tax revenues and government consumption. By construction, $Y_{t,i}(\theta^t) \leq Z_t(\theta^t)$ for all i. ■

For the preceding theorem to hold, the number of subperiods I in the game with one-period debt may need to be higher than the corresponding number of subperiods with many-period debt, J. For instance, if $J = 1$ in the many-period debt game (or there is only one debt auction and one repayment in each period), it is not the case that this can always be replicated with one-period debt without having multiple debt auctions in each period. In a given period, one needs as many subperiods as the total debt divided by the maximum of per-period government spending and tax revenues. If the time period is taken to be a quarter, for the United States in 2000, this number is roughly thirteen—the number of weeks in a quarter. For reasonable specifications of the level of government debt and the length of the period, the number of required subperiods is not unity, but it also is not high.

The intuition behind Theorem 2 is straightfoward. Suppose an equilibrium of the many-period debt game has, in a given period, a high value of total government debt. If this debt is all long-term (say console debt), then at no time does the government need to have a large gross flow from the household sector to the government during an auction. For instance, if a high α is realized, the government needs only to have an inflow sufficient to cover the current primary deficit $(G_t - \tau_t L_t)$ plus interest payments on past debt. On the other hand, if $I = 1$ and all debt must be one-period debt, then the government needs to raise at the single auction enough to cover the current primary deficit and the entire value of the old debt. If after this inflow it defaults it gets a higher current benefit since defaulting on the entire debt after collecting enough to pay it off is a higher surprise lump sum tax than defaulting after collecting a lower amount. The key to Theorem 2 is to show that by breaking the auctions into a finite and uniformly bounded number of smaller auctions, this effect of one period debt is eliminated.

Theorems 1 and 2 create a simplification regarding the computation of equilibria. Specifically, one can simply assume the existence of one type of paper at the beginning of a period. The next theorem goes further and shows that, in some sense, one does not need to consider the timing of transfers between the government and households, but instead consider only the totals (over the period) $\{P_t(\theta^t), X_t(\theta^t), R_t(\theta^t)\}$. In particular, it shows there is a way to construct the timing of these transfers such that the government never wishes to deviate if and only if two inequalities hold in each period. These inequalities ensure (1) that the government does not to wish to deviate when setting taxes and (2) that the government does not wish to deviate if, after taxes and labor are determined, it has on hand the maximum of path tax revenues and path government spending.

Theorem 3. *Consider a sequence* $\{(C_t, L_t, G_t, \tau_t, P_t, X_t, R_t)(\theta^t)\}_{t=1}^{\infty}$ *which satisfies*

$$u'(C_t)(1 - \tau_t) = w'(L_t)$$

and

$$u'(C_t)P_t = \beta E_{\theta_{t+1}}[u'(C_{t+1})R_{t+1}|\alpha_t]$$

for all $t \geq 1$ *and histories* θ^t*. For* $I = \overline{P}/\underline{G}$*, this sequence corresponds to the outcome path* *of an equilibrium of the one-period bond game iff*

$$\sum_{s=t}^{\infty} \beta^{s-t}(u(C_s) - w(L_s) + v(L_t - C_t)) \geq \underline{V}(\alpha_t) \tag{14}$$

and

$$\sum_{s=t}^{\infty} \beta^{s-t}(u(C_s) - w(L_s) + v(L_t - C_t)) \geq \tag{15}$$
$$\max_{X \geq 0}[u(L_t - Z_t + X) - w(L_t) + v(Z_t - X, \alpha_t)]$$
$$+ \delta \sum_{\alpha'} \pi(\alpha_{t+1}|\alpha_t)\underline{V}(\alpha')$$

for all $t \geq 1$ *and histories* θ^t, *where* $Z_t(\theta^t) := \max\{\tau_t(\theta^t)L_t(\theta^t), Lt(\theta^t) - C_t(\theta^t)\}$.

Proof. Suppose conditions (14) and (15) are satisfied. That (14) is satisfied implies the government will not wish to deviate when setting taxes. That (15) is satisfied implies that for a given t and θ^t, the government can be trusted not to deviate if the amount of the consumption good it has on hand at any point in the period does not exceed the maximum of tax revenues and government spending $Z_t(\theta^t)$. As in the proof to Theorem 2, one can use a sequence of auctions and repayments (of uniformly bounded length) to keep the amount of the consumption good the government has on hand weakly below this amount.

Next suppose (14) is violated. It is immediate that no structure of auctions and repayments can keep the government from deviating when setting taxes. Finally, suppose (15) is violated. Since the government must have on hand at least $Z_t(\theta_t)$ at some point regardless of the timing of auctions and repayments, it will always find it optimal to deviate at that point. ■

7. Conclusion

As stated in the introduction, large debt rollovers have been suggested as a potential culprit in several recent financial crises. This paper presents a plausible economic environment where this is impossible. It is important to be careful here, however. Nothing in this paper implies that the crises themselves should not have happened. The games presented here, with and without long-term debt, allow a large variety of equilibria, including sunspot equilibria which can be interpreted as allowing crises. What this paper shows is that such equilibria exist with and without long-term debt.

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