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A GENERALIZED VARIANCE BOUNDS TEST

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ABSTRACT

This paper derives a variance bounds test for a broad class of linear rational expectations models. According to this test, if observed data accord with the model, then a weighted sum of autocovariances of the covariance-stationary components of the endogenous state variables should be nonnegative. The new test reinterprets West's (1986) variance bounds test and extends its applicability by not requiring observable exogenous state variables, covariance-stationary exogenous or endogenous state variables, or a zero initial value for the endogenous state variable. The paper also discusses the possibility of the new test's application to nonlinear models.

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1. Introduction

West (1986) derived a variance bounds test for a linear rational expectations (LRE) version of the production smoothing/buffer stock model of inventories. He hypothesized that the covariance-stationary components of production, sales, and inventories are consistent with the optimal policy of the underlying dynamic maximization problem. Then he showed that the unconditional expectation of the difference between the value of the objective function under the optimal policy and its value under a policy with inventories identically zero is equal to a weighted sum of variances and covariances of production, sales, and inventories. Thus, he tested whether the nonnegativity of this sum is satisfied by the covariance-stationary components of the observed data.

My purpose here is to reinterpret and generalize West's variance bounds test for a broad class of LRE models. I show that the unconditional expectation of the difference between the values of the objective function of the underlying dynamic problem under an optimal policy and under any feasible policy, such that the difference between the corresponding state paths is a covariance-stationary process, is a weighted sum of autocovariances of that process. Thus, this sum must be nonnegative for any covariance-stationary difference between an optimal state path and a feasible state path. Two implications of this result stand out. One is that, if observed data accord with the optimal policy (that is, the theory), then their covariance-stationary components should satisfy the above condition. For, as shown below, the nonstationary components of the endogenous state variables qualify as the state path associated with a feasible policy. This is the reinterpretation result. The other noteworthy implication is that, unlike West's condition, the one derived here does not require observable exogenous state variables, covariance-stationary exogenous or endogenous state variables, or a zero initial value for the endogenous state variable. This generalization result is a consequence of the fact that the condition derived here exploits some of the other necessary conditions while West's does not.

Three sections follow. Section 2 sets up a general LRE model and reviews the standard necessary conditions for its solution. Section 3 derives the new condition, interprets it, and discusses its extension to nonlinear rational expectations models. Section 4 illustrates the economic importance of the new condition in the context of West's inventory model. All proofs are in an appendix.

2. A General Linear Rational Expectations Model

Let $\{\xi(t):t\in\mathbf{N}\}$, $\mathbf{N} = \{0,\pm1,\ldots\}$, be a stochastic process on a probability space $(\underline{0},\Omega,P)$, where $\xi(t)$ is an $(n_\xi\times 1)$-dimensional vector of exogenous state variables at the beginning of period t. Also, let Ω_t be the σ-algebra generated by the sequence of random variables $(\ldots,\xi(t-1),\xi(t))$, $t \in \mathbf{N}$. The term Ω_t represents the information available to the system at the beginning of period t. Clearly, $\Omega_t \subset \Omega_{t+1} \subset \Omega$, $\forall\ t \in \mathbf{N}$. The term $E(\cdot)$ denotes the unconditional expectations operator with respect to P. That is, for any integrable function $(\cdot)$ with respect to P, $E(\cdot) = \int_{\underline{0}} (\cdot)\, dP = \int_{\underline{0}} (\cdot)P\,(d\omega)$. The term $E_t(\cdot)$ denotes the conditional expectations operator with respect to P, given Ω_t. That is, for any integrable function $(\cdot)$ with respect to P such that $E(\cdot) < \infty$, $\int_A (\cdot)P\,(d\omega) = \int_A E_t(\cdot)P\,(d\omega)$, $\forall\ A \in \Omega_t$. The $\{\xi(t): t\in\{\tau,\tau+1,\ldots\}, \tau\in\mathbf{N}\}$ process takes values in ℓ_τ^ξ, that is, the space of sequences $\xi = [\xi(\tau),\xi(\tau+1),\ldots]$, $\tilde{\xi} = [\tilde{\xi}(\tau),\tilde{\xi}(\tau+1),\ldots]$, and so on, such that

$$\sum_{t=\tau}^{\infty} \beta^{t-\tau} E\xi(t)'\tilde{\xi}(t) < \infty$$

where $\beta \in (0,1)$ and is the discount factor in all periods. Also, let $x(t)$ be an $(n_x\times 1)$-dimensional vector of endogenous state variables at the beginning of period t. Then a variety of LRE models can be stated as a problem (P) of this form:[1]

$$(1) \qquad \max_{\{u(t)\}_{t=\tau}^{\infty}} \ \lim_{T\to\infty} E_\tau \sum_{t=\tau}^{T} \beta^{t-\tau} f[\xi(t),x(t),u(t)]$$

subject to the following:

$$(2) \qquad u(t) \text{ is } \Omega_t\text{-measurable}$$

$$(3) \qquad x(t+1) = g[\xi(t),x(t),u(t)]$$

$$(4) \qquad x(\tau) = \bar{x} \text{ (given)}$$

$$(5) \qquad \{x(t)\}_{t=\tau}^{\infty} \in \ell_\tau^x$$

where $u(t)$ is an $(n_u\times 1)$-dimensional vector of control variables in period t,

$$(6) \qquad f|^t = \begin{bmatrix} \phi_\xi \\ \phi_x \\ \phi_u \end{bmatrix}' \begin{bmatrix} \xi(t) \\ x(t) \\ u(t) \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} \xi(t) \\ x(t) \\ u(t) \end{bmatrix}' \begin{bmatrix} \phi_{\xi\xi} & \phi_{\xi x} & \phi_{\xi u} \\ \phi_{x\xi} & \phi_{xx} & \phi_{xu} \\ \phi_{u\xi} & \phi_{ux} & \phi_{uu} \end{bmatrix} \begin{bmatrix} \xi(t) \\ x(t) \\ u(t) \end{bmatrix},$$

ϕ_i and ϕ_{ij} are appropriately dimensioned vectors and matrices, respectively, such that

(7) $$\phi'_{ij} = \phi_{ji}$$

(8) $$g|^t = \gamma_{x\xi}\xi(t) + \gamma_{xx}x(t) + \gamma_{xu}u(t),$$

γ_{xi} are appropriately dimensioned matrices, such that there exists an $(n_u \times n_x)$-dimensional matrix δ_{ux} with the property

(9) $$\delta_{ux}\gamma_{xu} = I$$

and $\pounds_\tau^x$ is the space of sequences $x = (x(\tau), x(\tau+1), \ldots)$, $\tilde{x} = (\tilde{x}(\tau), \tilde{x}(\tau+1), \ldots)$, and so on, such that

$$\sum_{t=\tau}^{\infty} \beta^{t-\tau} E x(t)'\tilde{x}(t) < \infty.$$

This formulation implies that $x(t+1)$ is Ω_t-measurable $\forall\ t \in \{\tau, \tau+1, \ldots\}$. Thus, decisions at time t depend only on the history of the $\{\xi(t): t\in\mathbf{N}\}$ process and $\bar{x}$.

Note that no a priori curvature restrictions have been imposed on $f|^t$. Also, no explicit law of motion for the $\{\xi(t): t\in\mathbf{N}\}$ process has been postulated. But condition (5), that $\{x(t)\}_{t=\tau}^{\infty} \in \pounds_\tau^x$, can be relaxed. All that is necessary here is that $\beta^{T-\tau}|E_\tau x(T)'x(T)| \to 0$ as $T \to \infty$, $\forall\ \bar{x} \in \mathbf{R}^n x$. Moreover, (9) can be eliminated. Its implication is, of course, that the system has, at most, n_x controls. Effectively, this excludes all these models which have a solution that cannot be characterized by Euler conditions.

Now, for convenience, I will transform (P) as follows:

Fact 1. Given (7) and (9), problem (P) is equivalent to this problem (P'):

(10) $$\max_{\{x(t+1)\}_{t=\tau}^{\infty}} \lim_{T\to\infty} E_{\tau} \sum_{t=\tau}^{T} \beta^{t-\tau} h[\xi(t),x(t),v(t)]$$

subject to the following:

(11) $x(t+1)$ is Ω_t-measurable

(12) $v(t) = x(t+1) - x(t)$

(13) $x(\tau) = \bar{x}$

(14) $\{x(t)\}_{t=\tau}^{\infty} \in \pounds_{\tau}^{x}$

where

(15) $$h|^{t} = \begin{bmatrix} k \\ \ell \\ m \end{bmatrix}' \begin{bmatrix} \xi(t) \\ x(t) \\ v(t) \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} \xi(t) \\ x(t) \\ v(t) \end{bmatrix}' \begin{bmatrix} N & U & P \\ U' & Q & R \\ P' & R & S \end{bmatrix} \begin{bmatrix} \xi(t) \\ x(t) \\ v(t) \end{bmatrix}$$

(16) $N' = N$, $Q' = Q$, and $S' = S$

$$\ell = \phi_x + (I-\gamma_{xx})'\delta'_{ux}\phi_u$$

$$m = \delta'_{ux}\phi_u$$

$$U = \phi_{\xi x} - \gamma'_{x\xi}\delta'_{ux} + \phi_{\xi x}\delta_{ux}(I-\gamma_{xx}) - \gamma'_{x\xi}\delta'_{ux}\phi_{uu}\delta_{ux}(I-\gamma_{xx})$$

$$P = \phi_{\xi u}\delta_{ux} - \gamma'_{x\xi}\delta_{ux}\phi_{uu}\delta_{ux}$$

$$Q = \phi_{xx} + (I-\gamma_{xx})'\delta'_{ux}\phi_{ux} + \phi_{xu}\delta_{ux}(I-\gamma_{xx})$$
$$+ (I-\gamma_{xx})'\delta'_{ux}\phi_{uu}\delta_{ux}(I-\gamma_{xx})$$

$$R = \phi_{xu}\delta_{ux} + (I-\gamma_{xx})'\delta'_{ux}\phi_{uu}\delta_{ux}$$

$$S = \delta'_{ux}\phi_{uu}\delta_{ux}.$$

Then this is well known:

Fact 2. If $\{x(t+1)\}_{t=\tau}^{\infty}$ is an optimal policy for (P), then these conditions must hold:

Euler condition

$$(17) \qquad (S-R)E_t x(t+2) - [Q-(R+R')+(1+\beta^{-1})S]E_t x(t+1) + \beta^{-1}(S-R)'E_t x(t) = \ell + (\beta^{-1}-1)m + (U-P)'E_t\xi(t+1) + \beta^{-1}P'E_t\xi(t)$$

$\forall\ t \in \{\tau,\tau+1,\ldots\}$.

Legendre condition

(18) $Q - (R+R') + (1+\beta^{-1})S$ is negative semidefinite.

3. The New Condition

Now I can state and prove the necessity of the new condition.

Lemma 1. If

$$(19) \qquad \delta(t) = x^{+}(t) - x^{-}(t)$$

where $\{x^{+}(t+1)\}_{t=\tau}^{\infty}$ is an optimal policy for (P) and $\{x^{-}(t+1)\}_{t=\tau}^{\infty}$ is any feasible policy for (P), that is, satisfies (11)-(14), then the following condition must hold:

(20) $$\lim_{T\to\infty} E \sum_{t=\tau}^{T} \beta^{t-\tau}\Big[2\delta(t+1)'(S-R)\delta(t+2) + \delta(t+1)'\big(Q-R-R'+(1+\beta^{-1})S\big)\delta(t+1)\Big] \le 0.$$

(The proof is in the appendix.)

Then this condition follows:

<u>Theorem</u>. If $\delta(t)$ is defined as in Lemma 1 and $\{\delta(t):t\in\mathbf{N}\}$ is a covariance-stationary process, then

(21) $$E\Big[2\delta(\tau+1)'(S-R)\delta(\tau+2) + \delta(\tau+1)'\big(Q-(R+R')+(1+\beta^{-1})S\big)\delta(\tau+1)\Big] \le 0$$

$$\forall\ \tau \in \mathbf{N}.$$

Several comments are in order. First, condition (21) is indeed new. In particular, it is not the other second-order (Legendre) condition or implied by that condition. Actually, (21) has an interesting economic interpretation. From the proof of Lemma 1 (in the appendix) it can be shown that

$$\tfrac{1}{2}\beta^{t+1-\tau}\ E_{\tau}\Big[2\delta(t+1)'(S-R)\delta(t+1) + \delta(t+1)'(Q-R-R'+S+\beta^{-1}S)\delta(t+1)\Big]$$

is the conditional expectation of the benefit associated with any deviation $\{\delta(t)\}_{t=\tau}^{\infty}$ from the optimal plan in period t. Thus, (21) simply states that the expectation of the benefit associated with any covariance-stationary deviation $\{\delta(t)\}_{t=\tau}^{\infty}$ from the optimal plan in any period should be nonpositive.

Second, what makes this result useful is that if observed data accord with the optimal policy, then their deviations from their nonstationary components should satisfy (21), for these nonstationary components trivially satisfy all the requirements for a feasible solution (as I show below). Or if the maintained hypothesis is that the covariance-stationary components of the endogenous variables are generated by the optimal solution of the model and a zero feasible solution is meaningful (as in West's inventory model), then, again, (21) should be satisfied by the covariance-stationary component of the endogenous state variables. Thus, (21) provides a natural and easily implementable test for the validity of the hypothesis that observed data accord with the optimal solution, a test that does not require strong curvature restrictions or a specification of the law of motion of the exogenous state variables. Third, tracing the steps of the proof of Lemma 1 will easily verify that not imposing (9) produces a similar result.

Also, when h is twice differentiable but not necessarily quadratic and a transversality-like condition holds, (20)'s counterpart is

$$\lim_{T\to\infty} E_t \sum_{t=\tau}^{T-1} \beta^{t-\tau}\Big[\delta(t+1)[\nabla_{xv}h^{\theta}|^{t+1} - \nabla_{vv}h^{\theta}|^{t+1}]\delta(t+2)$$
$$+\delta(t+1)'[\nabla_{xx}h^{\theta}|^{t+1} - \nabla_{xv}h^{\theta}|^{t+1}$$
$$-\nabla_{vx}h^{\theta}|^{t+1} + (1+\beta^{-1})\nabla_{vv}h^{\theta}|^{t+1}]\delta(t+1)\Big] \le 0$$

where

$$h^{\theta}|^{t+1} = h\big[\xi(t+1), x^{\theta}(t+1), v^{\theta}(t+1)\big]$$

$$x^{\theta}(t+1) = \theta x^{+}(t+1) + (1-\theta)x^{-}(t+1)$$

and

$$v^{\theta}(t+1) = \theta v^{+}(t+1) + (1-\theta)v^{-}(t+1)$$

for $\theta \in (0,1)$. To reduce this condition to something like (21), $E[\nabla_{xv}h^{\theta}|^{t+1} - \nabla_{vv}h^{\theta}|^{t+1}]$ and $E[\nabla_{xx}h^{\theta}|^{t+1} - \nabla_{xv}h^{\theta}|^{t+1} - \nabla_{vx}h^{\theta}|^{t+1} + (1+\beta^{-1})\nabla_{vv}h^{\theta}|^{t+1}]$ must be constant matrices. This may be true after an appropriate transformation (for example, after multiplication by a positive random variable) if $\{x^{\theta}(t)\}_{t=\tau}^{\infty}$ is a stationary or steady state and $\{x^{-}(t+1)\}_{t=\tau}^{\infty}$ is obtained as $(1-\theta)^{-1}x^{\theta}(t+1) - \theta(1-\theta)^{-1}x^{+}(t+1)$ for an appropriately chosen θ. Then note that $\delta(t+1) = x^{+}(t+1) - x^{-}(t+1) = (1-\theta)^{-1}[x^{+}(t+1) - x^{\theta}(t+1)]$ implies that (21)'s counterpart is effectively independent of θ. So to evaluate the new condition, all that is needed is the covariance-stationary deviation from the steady state.

It remains to show how to construct observable covariance-stationary $\{\delta(t) : t \in \mathbf{N}\}$ processes and thereby check for (21).

Let $\Psi(t) = \xi(t) - E\xi(t)$. Suppose that

(22) $$\Psi(t+1) = A\Psi(t) + \varepsilon(t+1)$$

where

(23) $$\{z \in \mathbb{C} : \det(I - Az) = 0\} \cap \{z \in \mathbb{C} : |z| < 1\} = \emptyset$$

(24) $$\varepsilon(t) \sim N(0, \Sigma), \quad \forall\, t \in \mathbf{N}$$

(25) $$E\varepsilon(t)'\varepsilon(t') = 0, \quad \forall\, t \neq t'.$$

Then $\{\Psi(t) : t \in \mathbf{N}\}$ is a covariance-stationary process. Assume that

(26) $$\det(S - R) \neq 0$$

and let $J_1(J_2)$ be a Jordan matrix with the eigenvalues of

$$E(z) = Iz^2 - (S-R)^{-1}(Q-R-R'+S+\beta^{-1}S)z + \beta^{-1}(S-R)^{-1}(S-R)'$$

with modulus less (greater) than $\beta^{-\frac{1}{2}}$. Also, let $H_1(H_2)$ be the matrix with the eigenvectors and the generalized eigenvectors of $E(z)$ corresponding to $J_1(J_2)$, and add these regularity restrictions:

(27) $$\{z\in ¢:\det[E(z)]=0\} \cap \{z\in ¢:|z|=\beta^{-\frac{1}{2}}\} = \emptyset$$

(28) $$\text{rank } (H_i) = n_x, \quad i = 1, 2.$$

Then the policy for (17) subject to (4) and (5) is given by[2]

(29) $$x^+(t+1) = K_1x^+(t) + M\Psi(t) + N(t), \quad x^+(\tau) = \bar{x}$$

where

$$K_i = H_iJ_iH_i^{-1}$$

$$-M = K_2^{-1}(S-R)^{-1}\beta^{-1}P' + \sum_{j=1}^{\infty} K_2^{-j}[(S-R)^{-1}(U-P)'+K_2^{-1}(S-R)^{-1}\beta^{-1}P']A^j$$

$$-N(t) = (I-K_2^{-1})^{-1}K_2^{-1}(S-R)^{-1}[\ell+(\beta^{-1}-1)m] + K_2^{-1}(S-R)^{-1}\beta^{-1}P'E\xi(t) + \sum_{j=1}^{\infty} K_2^{-j}[(S-R)^{-1}(U-P)'+K_2^{-1}(S-R)^{-1}\beta^{-1}P']E\xi(t+j).$$

Consider, now, the deterministic problem that results from (**P**) by substituting $E\xi(t)$ for $\xi(t)$ in (1) and (3). Under the regularity conditions mentioned earlier, any optimal policy for this problem should satisfy

$$(30) \qquad x^-(t+1) = K_1 x^-(t) + N(t), \quad x^-(\tau) = \bar{x} .$$

Obviously, $\{x^-(t+1)\}_{t=\tau}^{\infty}$ is a feasible policy for (**P**). This policy, sometimes referred to as the <u>open-loop policy</u>, and (30) imply that

$$(31) \qquad x^+(t+1) = K_1^{t-\tau+1}\bar{x} + \sum_{i=0}^{t-T} K_1^i [M\Psi(t-i)+N(t-i)]$$

$$(32) \qquad x^-(t+1) = K_1^{t-\tau+1}\bar{x} + \sum_{i=0}^{t-T} K_1^i N(t-i).$$

Hence,

$$(33) \qquad \delta(t) \equiv x^+(t) - x^-(t) = \sum_{i=0}^{t-T} K_1^i M\Psi(t-i).$$

Thus, since $\delta(t)$ can be obtained as the finite sum of covariance-stationary processes, it is itself covariance-stationary. This result can be easily extended to account for moving average components in the law of motion of the $\{\Psi(t):t\in\mathbf{N}\}$ process.

4. <u>Comparison With West's Variance Bounds Test</u>

Now I will illustrate the economic importance of the new condition in the context of West's inventory model. One way to look at West's model is to consider a firm that takes as given its sales of a single homogeneous good and seeks a production schedule that will minimize its expected discounted future stream of real costs:

$$(34) \qquad C[\{Q(t)\}_{t=0}^{\infty}, H(-1), 0]$$

$$= E_0 \sum_{t=0}^{\infty} \beta^t \{a_0[Q(t)-Q(t-1)]^2 + a_1[Q(t)]^2 + a_2[H(t)-a_3S(t+1)]^2\}$$

$$a_i \in \mathbf{R}, \quad i = 0, 1, 2, 3$$

subject to

$$(35) \qquad Q(t) = S(t) + H(t) - H(t-1)$$

$$(36) \qquad H(-1) = 0$$

where E_0 and β are as in section 2, $Q(t)$ is production in period t, $H(t)$ is inventories at the end of period t, and $S(t+1)$ is the covariance-stationary component of sales in period $t+1$. The term $a_0[Q(t)-Q(t-1)]^2$ represents adjustment costs brought about by changing production levels. The term $a_1[Q(t)]^2$ represents production costs, and the term $a_2[H(t)-a_3S(t+1)]^2$ represents inventory holding and backlog costs. In this model, firms hold inventories for two reasons: to smooth production in the face of randomly fluctuating sales and to avoid sales backlogs. Neither the cost minimization hypothesis nor any particular market structure hypothesis is crucial here.

4.1 Deriving West's Test

To derive West's variance bounds test, I hypothesize that the optimal production plan $\{Q^*(t)\}_{t=0}^{\infty}$ and its associated inventory plan $\{H^*(t)\}_{t=0}^{\infty}$ are covariance-stationary processes. The production plan $\{Q^0(t)\}_{t=0}^{\infty}$, where production is set equal to sales, is

(37) $$Q^0(t) = S(t), \quad \forall\, t \in \mathbf{N}_+$$

so that no inventories are held:

(38) $$H^0(t) = 0, \quad \forall\, t \in \mathbf{N}_+.$$

Since that production plan is feasible, this must be true:

(39) $$E(C[\{Q^0(t)\}_{t=0}^{\infty},H(-1),0]-C[\{Q^*(t)\}_{t=0}^{\infty},H(-1),0])$$

$$= E(E_0\sum_{t=0}^{\infty}\beta^t\{a_0[S(t)-S(t-1)]^2+a_1[S(t)]^2$$

$$+a_2[-a_3S(t+1)]^2\} - E_0\sum_{t=0}^{\infty}\beta^t\{a_0[Q^*(t)-Q^*(t-1)]^2$$

$$+a_1[Q^*(t)]^2+a_2[H^*(t)-a_3S(t+1)]^2\})$$

$$= \sum_{t=0}^{\infty}\beta^t(a_0\{E[S(t)-S(t-1)]^2-E[Q^*(t)-Q^*(t-1)]^2\}$$

$$+a_1\{E[S(t)]^2-E[Q(t)]^2\}-a_2E[H^*(t)]^2$$

$$+2a_2a_3E[H^*(t)S(t+1)])$$

$$= \sum_{t=0}^{\infty}\beta^t\{a_0[\mathrm{var}(\Delta S)-\mathrm{var}(\Delta Q^*)]+a_1[\mathrm{var}(S)-\mathrm{var}(Q^*)]$$

$$-a_2\mathrm{var}(H^*)+2a_2a_3\mathrm{cov}(H^*,S_{+1})\}$$

$$= (1-\beta)^{-1}\{a_0[\mathrm{var}(\Delta S)-\mathrm{var}(\Delta Q^*)]+a_1[\mathrm{var}(S)-\mathrm{var}(Q^*)]$$

$$-a_2\mathrm{var}(H^*)+2a_2a_3\mathrm{cov}(H^*,S_{+1})\}$$

$$\geq 0$$

where

$$\mathrm{var}(Q^*) = E\left[Q^*(t)\right]^2, \quad \forall\ t \in \mathbf{N}_+$$

$$\mathrm{var}(\Delta Q^*) = E\left[Q^*(t)-Q^*(t-1)\right]^2, \quad \forall\ t \in \mathbf{N}_+$$

$$\mathrm{cov}(Q^*,Q^*_{+1}) = E\left[Q^*(t)Q^*(t+1)\right], \quad \forall\ t \in \mathbf{N}_+$$

and so on. The first equality in (39) follows from (35)-(38). The second equality follows from the fact that $EE_0(\cdot) = E(\cdot)$. The third and fourth equalities in (39) follow from the assumed covariance-stationarity of $\{S(t)\}_{t=0}^{\infty}$, $\{Q^*(t)\}_{t=0}^{\infty}$, and $\{H^*(t)\}_{t=0}^{\infty}$. Finally, the inequality in (39) follows, simply, from the fact that $\{Q^0(t)\}_{t=0}^{\infty}$ is an optimal plan for (34)-(36), but a feasible plan for this problem. Thus, based on the hypothesis that the covariance-stationary component of observed production is part of the optimal plan $\{Q^0(t)\}_{t=0}^{\infty}$, West developed a simple test for (39). Based on this test, he rejected the model for a number of nondurable manufacturing industries "even though the model does well by traditional criteria" [West (1986, p. 374)].

To compare the new condition to (39), I must map West's model in the (**P'**) format. Let

$$\xi(t) = \left(S(t+1),S(t),S(t-1)\right)' \quad x(t) = \left(H(t-1),H(t-2)\right)'$$

$$N/2 = \begin{bmatrix} a_2a_3^2 & 0 & 0 \\ 0 & a_0+a_1 & -a_0 \\ 0 & 0 & a_0 \end{bmatrix} \qquad U/2 = \begin{bmatrix} -a_2a_3 & 0 \\ -a_0 & a_0 \\ a_0 & -a_0 \end{bmatrix}$$

$$P/2 = \begin{bmatrix} -a_2a_3 & 0 \\ a_0+a_1 & 0 \\ -a_0 & 0 \end{bmatrix} \qquad Q/2 = \begin{bmatrix} a_0+a_2 & -a_0 \\ -a_0 & a_0 \end{bmatrix}$$

$$R/2 = \begin{bmatrix} -a_0+a_2 & 0 \\ a_0 & 0 \end{bmatrix} \qquad S/2 = \begin{bmatrix} a_0+a_1+a_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Now the new condition (21) gives

$$E\left\{2\begin{bmatrix}\delta_1(t+1)\\ \delta_2(t+1)\end{bmatrix}'\begin{bmatrix}-(2a_0+a_1) & 0\\ a_0 & 0\end{bmatrix}\begin{bmatrix}\delta_1(t+1)\\ \delta_2(t+1)\end{bmatrix}\right.$$

$$+\begin{bmatrix}\delta_1(t+1)\\ \delta_2(t+1)\end{bmatrix}\begin{bmatrix}(4+\beta^{-1})a_0+(1-\beta^{-1})a_1+\beta^{-1}a_2 & -2a_0\\ -2a_0 & a_0\end{bmatrix}$$

$$\left.\times\begin{bmatrix}\delta_1(t+1)\\ \delta_2(t+1)\end{bmatrix}\right\}\geq 0$$

or

(40) $$E\{[(5+\beta^{-1})a_0+(1+\beta^{-1})a_1+\beta^{-1}a_2][H(t)]^2$$

$$-2(4a_0+a_1)H(t)H(t-1)+2a_0H(t)H(t+2)\}\geq 0.$$

Lemma 2 clarifies the relationship between (39) and (40):

Lemma 2. Given $H(-2)$, $H(-1)$, $H^{-}(0)$, ... = 0, (39) holds if and only if (40) holds.

(The proof is in the appendix.) Thus, (39) is essentially identical to (21) or (40) when it is applied to West's inventory model.

4.2 Relaxing West's Assumptions

I turn now to juxtaposing (39) to (21) or (40) when some of West's assumptions are relaxed.

Case 1. Unobserved Exogenous State Variables

Suppose that production technology is subject to a random shock such that production costs are given by

$$\gamma(t)Q(t)+a_1Q(t)^2$$

where $\{\gamma(\cdot)\}$ is a covariance-stationary process observed by the firm but not by the econometrician.[3] It is straightforward to show that in this case (39) should be changed to

(41) $$\{a_0[\text{var}(\Delta S)-\text{var}(\Delta Q)]+ \text{cov}(\gamma,\Delta H)+a_1[\text{var}(S)-\text{var}(Q)]$$
$$-a_2\text{var}(H)+2a_2a_3\text{cov}(H^*,S_{+1})\} \geq 0.$$

However, since $\text{cov}(\gamma,\Delta H)$ is not estimable, (41) becomes inoperative.[4] Clearly, no such problem arises with (40).

<u>Case 2</u>. Nonstationary Exogenous State Variables

Suppose that sales follow a nonstationary process such that there exists $\bar{S} > 0$ and $s \in (0,\beta^{-\frac{1}{2}})$:

$$E_tS(t+j) < Ss^j, \quad \forall\ t_j \in \mathbf{N}_+.$$

(See Figure 1.) Clearly, then,

$$E[S(t)]^2$$

is not independent of t and hence the third equality in (39) is violated. Nevertheless, (21) and (40) remain valid.

<u>Case 3</u>. Nonstationary Endogenous State Variables

Suppose that $a_0 = a_3 = 0$, $a_1 > 0$, and $-(\beta^{-\frac{1}{2}}-1)a_1 < a_2 \leq 0$. Further, suppose that sales are governed by $S(t) - \rho S(t-1) = \varepsilon_s(t)$ for $|\rho| < 1$ and $\{\varepsilon_s(\cdot)\}$ is a white noise process. Now, the Euler condition (17) reduces to

(42) $$E_tH(t+2) - (1+\beta^{-1}+a_1^{-1}a_2)E_tH(t) + \beta^{-1}E_tH(t-1)$$
$$= \beta^{-1}E_tS(t) - E_tS(t+1).$$

The characteristic polynomial associated with (42) is

$$\lambda^2 - (1+\beta^{-1}+ a_1^{-1}a_2)\lambda + \beta^{-1}.$$

The smallest modulus root of this polynomial is $\lambda \in [1,\beta^{-\frac{1}{2}})$. (See Figure 2.) Therefore, (18) and (26)-(28) are satisfied and the unique solution to the Euler condition is

$$H(t) = \lambda H(t-1) - (1-\beta\lambda\rho)^{-1}\lambda(1-\beta\rho)S(t).$$

Clearly, $\{H(t)\}_{t=0}^{\infty}$ is not covariance-stationary. Thus, the third equality in (39) is violated. But (21) and its specialization, (40), remain valid.

Case 4. A Zero Initial Value for the Endogenous Variable

Clearly, if $H(-1) \neq 0$, then (38) should be replaced by

$$H^0(t) = H(-1), \quad \forall\ t \in \mathbf{N}_+$$

which violates the first equality in (39). But, again, (21) and (40) remain valid even when $H(-1) \neq 0$.

These four cases are, of course, possible extensions of West's inventory model, extensions in which (39) is no longer valid but (21) and (40) are. Nevertheless, (21) clearly applies to all linear rational expectations models and to several non-linear ones. The applicability of (21), that is, does not rely on the existence of a zero feasible solution.

Notes

[1]For examples of LRE models, see Hansen and Sargent (1981) and Sargent (1982, 1985).

[2]For a proof of this, see Kollintzas (1986a,b). The regularity conditions (26), (27), and (28) are discussed in Kollintzas (1985, 1986b). Condition (27) is somewhat stronger than necessary. These conditions are not sufficient for (29) to be a solution to (**P**).

[3]This cost shock plays a major role in a variety of inventory models. See, for example, Eichenbaum (1984) and Kollintzas and Husted (1984).

[4]A similar result occurs when inventory holding costs are subject to a random shock as in the studies mentioned in note 3.

Appendix
Proofs of Lemmas

Proof of Lemma 1

Since $h|^t$ is quadratic, Taylor's theorem implies that

$$\text{(A1)} \qquad \Delta_\tau^T \equiv E_\tau \sum_{t=\tau}^{T} \beta^{t-\tau}(h^+|^t - h^-|^t)$$

$$= E_\tau \sum_{t=\tau}^{T} \beta^{t-\tau}\{\nabla_x h^+|^t[x^+(t)-x^-(t)]+\nabla_v h^+|^t[v^+(t)-v^-(t)]$$

$$-\tfrac{1}{2}[x^+(t)-x^-(t)]'\nabla_{xx}h|^t[x^+(t)-x^-(t)]$$

$$-\tfrac{1}{2}[x^+(t)-x^-(t)]'\nabla_{xv}h|^t[v^+(t)-v^-(t)]$$

$$-\tfrac{1}{2}[v^+(t)-v^-(t)]'\nabla_{vx}h|^t[x^+(t)-x^-(t)]$$

$$-\tfrac{1}{2}[v^+(t)-v^-(t)]'\nabla_{vv}h|^t[v^+(t)-v^-(t)]\}$$

where $\nabla_x h|^t$ stands for the gradient of h with respect to x evaluated at $h[\xi(t),x(t),v(t)]$, $\nabla_{xx}h|^t$ stands for the Hessian of h with respect to x evaluated at $h[\xi(t),x(t),v(t)]$, and so on.

Since $v(t) = x(t+1) - x(t)$ and $\delta(t) = x^+(t) - x^-(t)$,

$$\text{(A2)} \qquad \Delta_\tau^T = E_\tau \sum_{t=\tau}^{T} \beta^{t-\tau}\{[\nabla_x h^+|^t - \nabla_v h^+|^t]\delta(t) + \nabla_v h^+|^t \delta(t+1)$$

$$-\tfrac{1}{2}\delta(t)'[\nabla_{xx}h|^t - \nabla_{xv}h|^t - \nabla_{vx}h|^t + \nabla_{vv}h|^t]\delta(t)$$

$$-\tfrac{1}{2}\delta(t)'[\nabla_{xv}h|^t - \nabla_{vv}h|^t]\delta(t+1)$$

$$-\tfrac{1}{2}\delta(t+1)'[\nabla_{vx}h|^t - \nabla_{vv}h|^t]\delta(t+1)$$

$$-\tfrac{1}{2}\delta(t+1)'\nabla_{vv}h|^t\delta(t+1)\}.$$

Since both $\{x^+(t+1)\}_{t=\tau}^{\infty}$ and $\{x^-(t+1)\}_{t=\tau}^{\infty}$ are assumed to be feasible, (13) implies that $\delta(\tau) = 0$. Then a change of time indexes produces this:

(A3)
$$\begin{aligned}\Delta_\tau^T = E_\tau\Big(\sum_{t=\tau}^{T-1}\beta^{t+1-\tau}\{[\nabla_x h^+|^{t+1} - \nabla_v h^+|^{t+1}]\delta(t+1) \\ -\tfrac{1}{2}\delta(t+1)'[\nabla_{xx}h|^{t+1} - \nabla_{xv}h|^{t+1} \\ -\nabla_{vx}h|^{t+1} + \nabla_{vv}h|^{t+1}]\delta(t+1) \\ -\delta(t+1)'[\nabla_{xv}h|^{t+1} - \nabla_{vv}h|^{t+1}]\delta(t+2)\} \\ +\sum_{t=\tau}^{T}\beta^{t-\tau}\{\nabla_v h^+|^t\delta(t+1) - \tfrac{1}{2}\delta(t+1)'h_{vv}|^t\delta(t+1)\}\Big) \\ = E_\tau\Big(\beta\sum_{t=\tau}^{T-1}\beta^{t-\tau}[\beta^{-1}\nabla_v h^+|^t + \nabla_x h^+|^{t+1} - \nabla_v h^+|^{t+1}]\delta(t+1) \\ +\beta^{t-\tau}\{\nabla_v h^+|^T\delta(T+1) - \tfrac{1}{2}\delta(T+1)'h_{vv}|^T\delta(T+1)\} \\ -\tfrac{1}{2}\beta\sum_{t=\tau}^{T-1}\beta^{t-\tau}\{2\delta(t+1)'[\nabla_{xv}h|^{t+1} - \nabla_{vv}h|^{t+1}]\delta(t+2) \\ +\delta(t+1)'[\nabla_{xx}h|^{t+1} - \nabla_{xv}h|^{t+1} - \nabla_{vx}h|^{t+1} \\ +(1+\beta^{-1})\nabla_{vv}h|^{t+1}]\delta(t+1)\}\Big)\end{aligned}$$

or, in the (P') notation,

(A4)
$$\begin{aligned}\Delta_\tau^T = E_\tau\Big(\beta\sum_{t=\tau}^{T-1}\beta^{t-\tau}[(S-R)x^+(t+2) - (Q-R-R'+S+\beta^{-1}S)x^+(t+1) \\ +\beta^{-1}(S-R)'x^+(t) - \ell - (\beta^{-1}-1)m - (U-P)'\xi(t+1) \\ -\beta^{-1}P'\xi(t)]'\delta(t+1) \\ +\beta^{T-\tau}\{[Sx^+(T+1) + (R-S)'x^+(T) + m + P'\xi(T)]'\delta(T+1) \\ -\tfrac{1}{2}\delta(T+1)'S\delta(T+1)\}\end{aligned}$$

$$-\tfrac{1}{2}\beta\sum_{t=\tau}^{T-1}\beta^{t-\tau}\{2\delta(t+1)'(S-R)\delta(t+2)$$
$$+\delta(t+1)'(Q-R-R'+S+\beta^{-1}S)\delta(t+1)\}\}.$$

Now, $\Omega_t \subset \Omega_{t+1}$, $\forall\ t \in \mathbf{N}$, so $E_\tau(\cdot) = E_\tau(E_t(\cdot))$, $\forall\ t \in \{\tau,\tau+1,\ldots\}$. [See, for example, Billingsley (1986, Theorem 34.4).] Since $\{x^+(t+1)\}_{t=\tau}^{\infty}$ and $\{x^-(t+1)\}_{t=\tau}^{\infty}$ are assumed to be Ω_t-measurable, (12) implies that $\delta(t+1)$ is Ω_t-measurable. Therefore, $E_t\delta(t+1) = \delta(t+1)$ and $E_t[(\cdot)\delta(t+1)] = [E_t(\cdot)]\delta(t+1)$. [See, for example, Billingsley (1986, Theorem 34.3).] These facts imply that

$$\text{(A5)}\qquad E_\tau[(S-R)x^+(t+2)-(Q-R-R'+S+\beta^{-1}S)x^+(t+1)+\beta^{-1}(S-R)'x^+(t)-\ell$$
$$-\{(\beta^{-1}-1)m-(U-P)'\}(t+1)-\beta^{-1}P'\xi(t)]'\delta(t+1)$$
$$= E_\tau\{[(S-R)E_tx^+(t+2)-(Q-R-R'+S+\beta^{-1}S)E_tx^+(t+1)$$
$$+\beta^{-1}(S-R)'E_tx(t)-\ell+(\beta^{-1}-1)m-(U-P)'E_t\xi(t+1)$$
$$-\beta^{-1}P'E_t\xi(t)]'\delta(t+1)\}.$$

Therefore, since $\{x^+(t+1)\}_{t=\tau}^{\infty}$ is assumed to be an optimal policy for (**P**), (17) implies that the first term on the right side of (A4) is zero. Furthermore, since $(\{\xi(t)\}_{t=\tau}^{\infty},\{x^+(t)\}_{t=\tau}^{\infty})$, $(\{\xi(t)\}_{t=\tau}^{\infty},\{x^-(t)\}_{t=\tau}^{\infty}) \in \mathcal{L}_\tau^\xi \times \mathcal{L}_\tau^x$, each of the bilinear and quadratic forms of the second term on the right side of (A4) goes to zero as $T \to \infty$. Therefore,

$$\text{(A6)}\qquad \lim_{T\to\infty}\Delta_\tau^T = -\tfrac{1}{2}\beta\lim_{T\to\infty}E_\tau\sum_{t=\tau}^{T-1}\beta^{t-\tau}[2\delta(t+1)'(S-R)\delta(t+2)$$
$$+\delta(t+1)'(Q-R-R'+S+\beta^{-1}S)\delta(t+1)].$$

Now since $E(\cdot) = E(E_t(\cdot))$, $\forall\ t \in \mathbf{N}$, it follows immediately that if $\{x^+(t+1)\}_{t=\tau}^{\infty}$ is an optimal policy for (**P**) and $\{x^-(t+1)\}_{t=\tau}^{\infty}$ is any feasible policy for (**P**), then (20) must hold.

Q.E.D.

Proof of Lemma 2

Since $\{Q(t),S(t),H(t):t\in\mathbf{N}\}$ is a covariance-stationary process and $E[E_0(\cdot)] = E(\cdot)$,

$$\text{(A7)} \qquad \Phi = (1-\beta)^{-1}E\big(a_0\{[S(t)-S(t-1)]^2-[Q(t)-Q(t-1)]^2\}$$
$$+a_1\{[S(t)]^2-[Q(t)]^2\}-a_2[H(t)]^2$$
$$+2a_2a_3H(t)S(t+1)\big)$$

$$= EE_0\sum_{t=0}^{\infty}\beta^t\big(a_0\{[S(t)-S(t-1)]^2-[Q(t)-Q(t-1)]^2\}$$
$$+a_1\{[S(t)]^2-[Q(t)]^2\}-a_2[H(t)]^2$$
$$+2a_2a_3H(t)S(t+1)\big).$$

$$\equiv EE_0\sum_{t=0}^{\infty}\beta^t\begin{bmatrix} S(t+1) \\ S(t) \\ S(t-1) \\ H(t-1) \\ H(t-2) \\ H(t)-H(t-1) \\ H(t-1)-H(t-2) \end{bmatrix}'$$

$$\times\begin{bmatrix} a_2a_3^2 & 0 & 0 & -a_2a_3 & 0 & -a_2a_3 & 0 \\ & a_0+a_1 & -a_0 & -a_0 & a_0 & a_0+a_1 & 0 \\ & & a_0 & a_0 & -a_0 & -a_0 & 0 \\ & & & a_0+a_2 & -a_0 & -a_0+a_2 & 0 \\ & & & & a_0 & a_0 & 0 \\ & & & & & a_0+a_1+a_2 & 0 \\ & & & & & & 0 \end{bmatrix}$$

$$\times\begin{bmatrix} S(t+1) \\ S(t) \\ S(t-1) \\ H(t-1) \\ H(t-2) \\ H(t)-H(t-1) \\ H(t-1)-H(t-2) \end{bmatrix}$$

$$= EE_0 \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} \xi(t) \\ x^+(t) \\ v^+(t) \end{bmatrix}' \begin{bmatrix} N & U & P \\ U' & Q & R \\ P' & R' & S \end{bmatrix} \begin{bmatrix} \xi(t) \\ x^+(t) \\ v^+(t) \end{bmatrix}.$$

Also, as in the proof of Lemma 1, it follows that

$$\text{(A8)} \qquad X \equiv -\tfrac{1}{2}\beta(1-\beta^{-1})E\{[(5+\beta^{-1})a_0+(1+\beta^{-1})a_1+\beta^{-1}a_2][H(t)^2]$$
$$-2(4a_0+a_1)H(t)H(t-1)+2a_0H(t)H(t-2)\}$$

$$= E\left(E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{bmatrix} \xi(t) \\ x^+(t) \\ v^+(t) \end{bmatrix}' \begin{bmatrix} N & U & P \\ U' & Q & R \\ P' & R' & S \end{bmatrix} \begin{bmatrix} \xi(t) \\ x^+(t) \\ v^+(t) \end{bmatrix} - \begin{bmatrix} \xi(t) \\ x^-(t) \\ v^-(t) \end{bmatrix}' \begin{bmatrix} N & U & P \\ U' & Q & R \\ P' & R' & S \end{bmatrix} \begin{bmatrix} \xi(t) \\ x^-(t) \\ v^-(t) \end{bmatrix} \right\}\right).$$

Thus, if $H(-2) = H(-1) = H^-(\tau) = \ldots \equiv 0$, then $\Phi = X$. Hence, since $\beta \in (0,1)$, (39) holds if and only if (40) holds.

Q.E.D.

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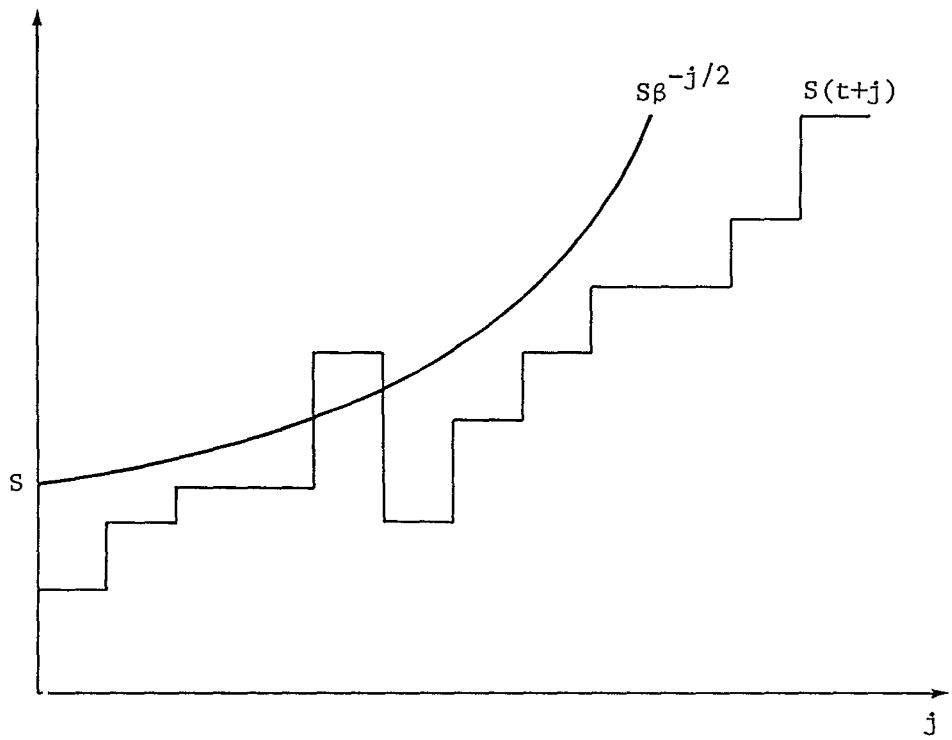

Figure 1

Nonstationary Exogenous State Variables
Allowed in (21)

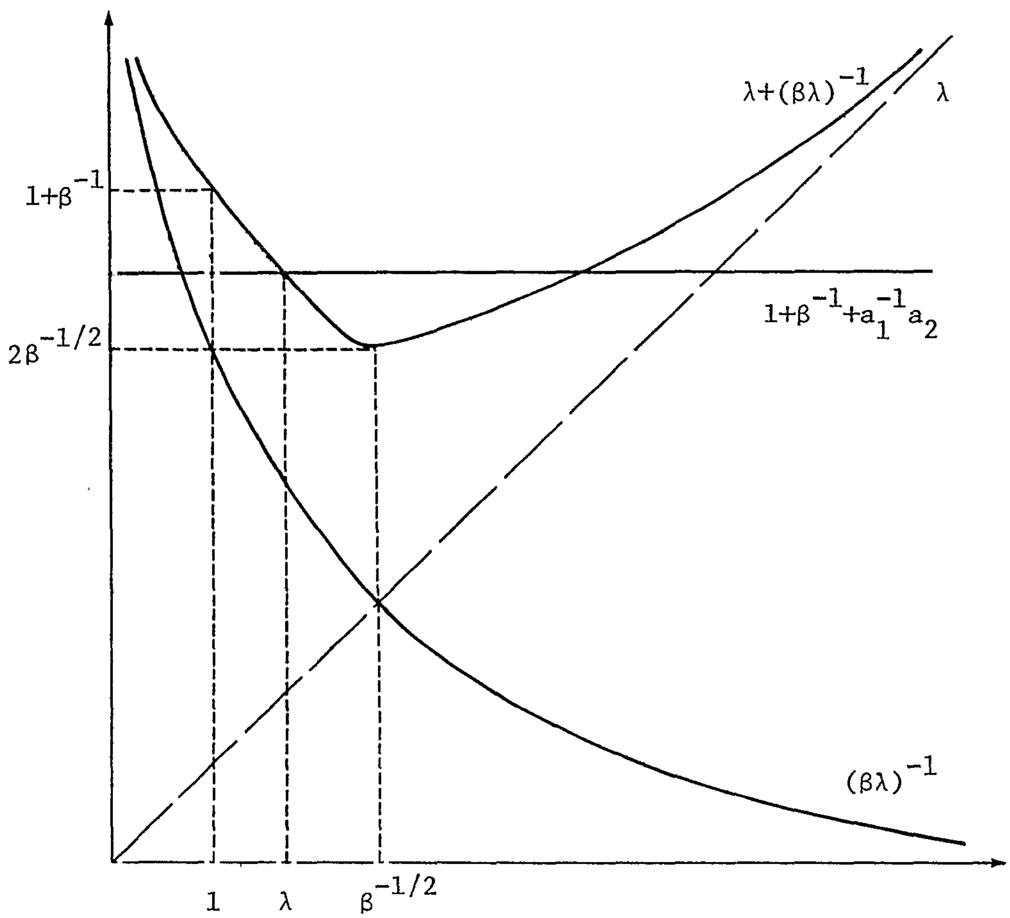

Figure 2

Nonstationary Endogenous State Variables
Allowed in (21)