Assortative Learning

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November 19, 2009

Motivation

Sorting and Turnover

- Sorting: High ability workers tend to sort into high productivity jobs: Positive Assortative Matching (PAM)
  $\Rightarrow$ Becker's (1973) theory of matching
- But, Becker is silent on turnover: job turnover tends to happen early in the life cycle
  $\Rightarrow$ Jovanovic (1979): canonical turnover model (learning)
- Assortative Learning: unified approach to sorting and job turnover:
  - Different learning rates across firms $\Rightarrow$ trade off wage vs. experimentation in better job (e.g., lower wage at top firm)
  - Is there sorting: Higher types $\Rightarrow$ in more productive firms?
  - Evolution of wages, turnover? Wage distribution?

Assortative Learning

- Like a two-armed bandit, but with:
  1. Large population – continuum of experimenters
  2. Correlated arms (general human capital)
  3. Endogenous payoffs (determined by equilibrium prices)
- Wage setting: spot market wages; no contingent contracts

Related literature

1. Labor-learning literature
   - Jovanovic (1979, 1984), Harris and Holmström (1982), Felli and Harris (1996), Moscarini (2005), Papageorgiou (2009)
2. Matching and Reputations
   - Anderson-Smith (2009): no PAM under SupM: set up of two-sided learning and symmetry $\Rightarrow$ no learning under PAM
3. Continuous time games
   - Sannikov (2007, 2008), Faingold and Sannikov (2007), Faingold (2007), Sannikov and Skrzypacz (2009)
4. Experimentation and bandit problem
   - Bergemann and Välimäki (1996), Bolton and Harris (1999), Keller and Rady (1999), Cripps et al. (2005)

Results

1. PAM unique equilibrium allocation under supermodularity, even with different learning rates across firms
2. Equilibrium efficient (despite incomplete markets/contracts)
3. Can account for increasing wage variance over life cycle; turnover and human capital accumulation
4. Theory: new no-deviation condition from sequential rationality (one-shot deviation principle) $\Rightarrow$ condition on second derivative of value function

Model setup

- Time is continuous, $t \in (-\infty, +\infty)$
- A unit measure of workers and a unit measure of firms
- Firms: infinitely lived, type $y \in \{H, L\}$, observable, and the fraction of H type firms is π
- Workers: type $x \in \{H, L\}$, not observable, both to firms and workers $\Rightarrow$ information is symmetric
- Birth and death of workers, both at exogenous rate δ
- A newborn worker is of type H with probability p_0 and of type L with probability $1 - p_0$
- Worker's entire output history is observable to all agents in the economy $\Rightarrow$ *common* belief about the worker type $p \in [0, 1]$: probability that $x = H$

Preferences and production

- Workers and firms are risk-neutral and discount future payoffs at rate $r > 0$
- Output is produced in pairs of one worker and one firm (x, y). Utility is perfectly transferable
- Expected output for each pair is denoted by μ_{xy}. We assume: $\mu_{Hy} \geq \mu_{Ly}, \forall y$ and $\mu_{xH} \geq \mu_{xL}, \forall x$
- Strict Supermodularity SupM (submodularity SubM with <):

$$\text{SupM: } \mu_{HH} + \mu_{LL} > \mu_{LH} + \mu_{HL}$$

Information

- Expected output is not perfectly observable, only the distorted variable (output) X is observed
- The realized cumulative output X_t is assumed to be a Brownian motion with drift μ_{xy} and common variance σ^2 (starting upon entry):

$$X_t = \mu_{xy} t + \sigma Z_t$$

- Both parties face the same information extraction problem

Equilibrium

- Denote expected values for firms and workers by $V_y, W_y(p)$ and wages by $w_y(p)$
- Spot market wages. Not condition on future actions/realiz.

Definition

In a (stationary) competitive equilibrium, there is a competitive wage schedule $w_y(p) = \mu_y(p) - rV_y$ for firm $y = H, L$ and worker p chooses firm y with the highest discounted present value. The market clears such that the measure of workers working in the L firm is $1 - \pi$ and the measure of workers working in the H firm is π.

Benchmark case: no learning

Claim

Given a distribution of $p, F(p)$. *Under SupM, PAM is the unique (stationary) competitive equilibrium allocation: H firms match with workers* $p \in [\underline{p}, 1]$, *L firms match with workers* $p \in [0, \underline{p})$, *where* $F(\underline{p}) = 1 - \pi$. *The opposite (NAM) holds under SubM.*

Belief updating

Lemma

(Belief Consistency) Consider any worker who works for firm y between t_0 and t_1. Given a prior $p_{t_0} \in (0,1)$, the posterior belief $(p_t)_{t_0<t\leq t_1}$ is consistent with the output process $(X_{y_t})_{t_0<t\leq t_1}$ if and only if it satisfies

$$dp_t = p_t(1-p_t)s_y d\bar{Z}_{y,t}$$

where

$$s_y = \frac{\mu_{Hy} - \mu_{Ly}}{\sigma},\ y = H, L$$

- Denote: $\Sigma_y(p) = \frac{1}{2}p^2(1-p)^2 s_y^2$

Value functions

- Worker's value function (from Ito's Lemma):

$$rW_y(p) = \mu_y(p) - rV_y + \Sigma_y(p) W_y^{''}(p) - \delta W_y(p)$$

  where $\mu_y(p) = p\mu_{Hy} + (1-p)\mu_{Ly}$
- Given linear output, learning value from option to switch y
- The general solution to this differential equation is:

$$W_y(p) = \frac{\mu_y(p) - rV_y}{r+\delta} + k_{y1} p^{1-\alpha_y}(1-p)^{\alpha_y} + k_{y2} p^{\alpha_y}(1-p)^{1-\alpha_y},$$

  where $\alpha_y = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta)}{s_y^2}} \geq 1.$

Equilibrium characterization

Value functions

1. For any possible cutoff $\underline{p}$:
   - Value-matching condition: $W_H(\underline{p}) = W_L(\underline{p})$
   - Smooth-pasting condition: $W'_H(\underline{p}) = W'_L(\underline{p})$

   On-the-equilibrium path conditions
2. Lemma 1: equilibrium value function W_y strictly increasing
3. Lemma 2: equilibrium value function W_y strictly convex
   From: positive option value of learning and linear pref.

Equilibrium characterization

No-deviation condition

Lemma

To deter possible deviations, a necessary condition is:

$$W_H''(\underline{p}) = W_L''(\underline{p}) \quad \textit{(No-deviation condition)}$$

for any possible cutoff $\underline{p}$.

- On equilibrium path, assume $p > \underline{p}$ match H, $p < \underline{p}, L$
- One-shot deviation: $p > \underline{p}$ worker with L for dt, then back H
- The value function for a deviator is:

$$\tilde{W}_L(p) = w_L(p)dt + e^{-(r+\delta)dt}[W_H(p) + \Sigma_L(p)W_H''(p)dt]$$

$$\lim_{dt \to 0} \frac{\tilde{W}_L(p) - W_H(p)}{dt} = w_L(p) - w_H(p) + [\Sigma_L(p) - \Sigma_H(p)]W_H''(p)$$

- Let $p \to \underline{p}$, then this is negative provided:

$$W_H''(\underline{p}) \leq W_L''(\underline{p})$$

Equilibrium characterization

Uniqueness result

Theorem

PAM is the unique stationary competitive equilibrium allocation under SupM. Likewise for NAM under SubM

- Cannot have p_1, p_2:

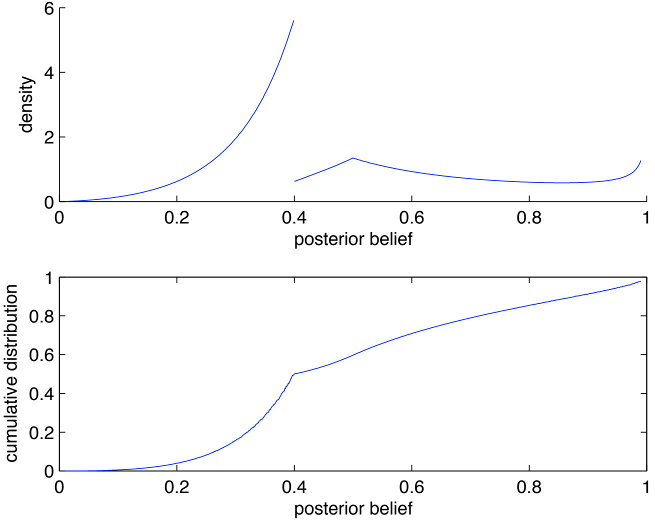
$$p \overset{L}{<} p_1 \quad p \overset{H}{\in} [p_1, p_2] \quad p \overset{L}{>} p_2$$

Equilibrium allocation and distribution

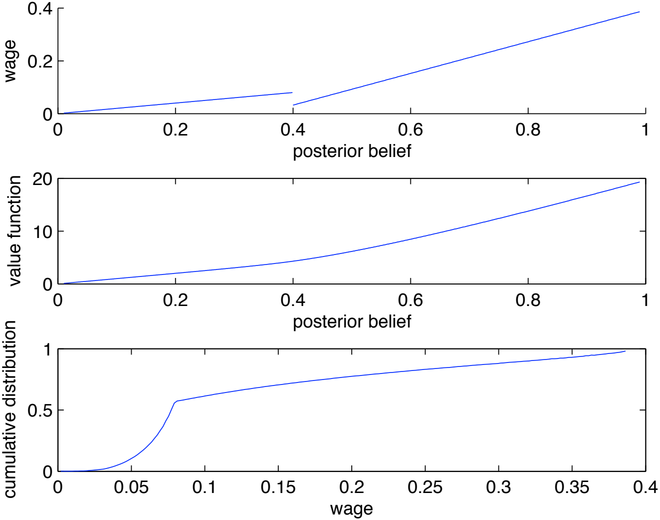
Ergodic distribution

Parameters: $s_H = 0.15, s_L = 0.05, p_0 = 0.5, \pi = 0.5, \delta = 0.01$.

Figure 2: Equilibrium Distribution of Posterior Beliefs



Equilibrium Payoffs, Value Functions



Surprising Implication of No-Deviation Condition

Firm-Dependent Volatility σ_y

- Existing setup:

$$X_t = \mu_{xy} t + \sigma \; Z_t$$

- *H* firms are superior in signal-to-noise ratio (from SupM):

$$s_H = \frac{\mu_{HH} - \mu_{LH}}{\sigma} > \quad \frac{\mu_{HL} - \mu_{LL}}{\sigma} = s_L,$$

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$$s_H = \frac{\mu_{HH} - \mu_{LH}}{\sigma_H} > < \frac{\mu_{HL} - \mu_{LL}}{\sigma_L} = s_L,$$

- Suppose instead that noise is firm-dependent: σ_y, then it is possible that $s_H < s_L$
- Note: we cannot have worker-dependent volatility σ_x from Girsanov's Theorem

Surprising Implication of No-Deviation Condition

Firm-Dependent Volatility σ_y

- Value function depends on s_y via $\Sigma_y = \frac{1}{2}p^2(1-p)^2 s_y^2$:

$$rW_y(p) = \mu_y(p) - rV_y + \Sigma_y(p)W_y''(p) - \delta W_y(p)$$

- Intuitively: W_H smaller than W_L?
- Inuition is Wrong:
  1. Wages are endogenous $\Rightarrow$ change as Σ_y changes
  2. No-deviation: $W_H'' = W_L''$

  $\Rightarrow$ Effect of learning is same in both firms irrespective of σ_y
- This result follows from sequential rationality + competitive price setting

The Planner's Problem

Proposition

The competitive equilibrium decentralizes the planner's solution that maximizes the aggregate flow of output.

- Surprising? Suppose $s_H^2 \to 0, s_L^2 \to \infty$
- Then: always allocate entrants to L firm to reveal type, even if not PAM
- But does not help efficiency, from martingale property

Labor Market Implications

Wage Variance over Life Cycle

Mean of posteriors:

$$\mathbb{E}p(t) = \int_0^{\underline{p}} p f_L^T(p,t)dp + \int_{\underline{p}}^1 p f_H^T(p,t)dp = p_0.$$

Our interest is with the variance of this distribution, which can be written as:

$$Var(p,t) = \int_0^{\underline{p}} p^2 f_L^T(p,t)dp + \int_{\underline{p}}^1 p^2 f_H^T(p,t)dp - p_0^2.$$

Proposition

The variance of beliefs, wages will eventually increase

- Standard learning model: wage variance decreases
- Evidence: variance over the life cycle increases and is concave (see e.g., Heathcoate, Violante and Perri 2009)

Labor Market Implications

Human Capital Accumulation

- In addition to learning unknown type, workers accumulate HC over life cycle
- Model prediction: wages of low types fall; counterfactual
- Assume: w.p. λ, a worker x becomes experienced and produces $\mu_{xy} + \xi_x$. The value functions are:

$$rW^e_y(p) = \mu_y(p) + \xi(p) - rV_y + \Sigma^e_y(p) W^{e\prime\prime}_y(p) - \delta W^e_y(p)$$
$$rW^u_{yy}(p) = \mu_y(p) - rV_y + \Sigma^u_y(p) W^{u\prime\prime}_{yy}(p) + \lambda W^e_y(p) - (\delta + \lambda) W^u_{yy}(p)$$
$$rW^u_{LH}(p) = \mu_L(p) - rV_L + \Sigma^u_L(p) W^{u\prime\prime}_{LH}(p) + \lambda W^e_H(p) - (\delta + \lambda) W^u_{LH}(p)$$

- Two cut-offs $\underline{p}^u, \underline{p}^e$ – need to show that $\underline{p}^u > \underline{p}^e$ given value functions

Proposition

Assume supermodularity and $\xi_H \simeq \xi_L$. Then $\underline{p}^e < \underline{p}^u$.

Labor Market Implications

Human Capital Accumulation

The expected tenure $\tau_y(p)$ satisfies the differential equation:

$$\Sigma_y(p)\tau_y''(p) - \delta p = -1,$$

with solutions (similar for $\tau_H^e, \tau_L^u, \tau_L^e$):

$$\tau_H^u(p) = \frac{1}{\delta}\left\{1 - \left(\frac{p}{\underline{p}^u}\right)^{1/2-\sqrt{1/4+2\delta/(s_H^u)^2}} \left(\frac{1-p}{1-\underline{p}^u}\right)^{1/2-\sqrt{1/4-2\delta/(s_H^u)^2}}\right\}$$

Proposition

(Tenure) Assume supermodularity and $\xi_H \simeq \xi_L$. Then, $\tau_L^u(p) > \tau_L^e(p)$ for $p < \underline{p}^e$ and $\tau_H^u(p) < \tau_H^e(p)$ for $p > \underline{p}^u$. For $p \in (\underline{p}^e, \underline{p}^u)$, there is a cutoff such that $\tau_L^u(p) < \tau_H^e(p)$ for p higher than this cutoff and $\tau_L^u(p) > \tau_H^e(p)$ for p smaller than this cutoff.

- Turnover very low p higher when e; for very high p, higher when u; intermediate depends on “closeness” of cutoff

Robustness

I. Generalized Lévy Processes

Conjecture

SupM $\Rightarrow$ PAM true for any Bayesian learning process

- From the Martingale Property; but need to solve $W(p)$
- Lévy process (compound Poisson): λ_{xy} arrival jumps, then

$$\begin{aligned}(r+\delta+[p\lambda_{Hy}+(1-p)\lambda_{Ly}])W_y(p) = \\ w_y(p)+[p\lambda_{Hy}+(1-p)\lambda_{Ly}]W_{y'}(p_h) \\ -p(1-p)(\lambda_{Hy}-\lambda_{Ly})W_y'(p)+\Sigma_y(p)W_y''(p)\end{aligned}$$

  where $p_h = \frac{p\lambda_{Hy}}{p\lambda_{Hy}+(1-p)\lambda_{Ly}}$, y' is firm that matches with p_h
- In the absence of jumps, the posterior follows:

$$dp = -p(1-p)(\lambda_{Hy}-\lambda_{Ly})dt + p(1-p)s_y d\bar{Z}$$

- Can solve ODE + No-deviation holds: $W_H''(\underline{p}) = W_L''(\underline{p})$

Proposition

Given the Lévy process, PAM is a stationary competitive equilibrium allocation under strict supermodularity.

Robustness

II. Non-Bayesian Updating

- Let belief updating: $dp = \lambda_y p dt$ for $p < 1$, and $dp = 0$ when $p = 0$. Then:

$$(r+\delta)W_y(p) = w_y(p) + \lambda_y p W_y'(p)$$

- We can solve the ODE. Equilibrium requires:

$$W_H(\underline{p}) = W_L(\underline{p}) \qquad \text{(Value Matching)}$$
$$W_H'(\underline{p}) = W_L'(\underline{p}) \qquad \text{(Smooth-pasting)}$$

- If $\lambda_L > \lambda_H$, PAM requires that

$$\frac{\mu_{LL} - rV_L}{r+\delta} > \frac{\mu_{LH} - rV_H}{r+\delta}$$
$$\frac{\lambda_H - \lambda_L}{r+\delta} \frac{\Delta_H}{r+\delta-\lambda_H}[\underline{p} - (\underline{p})^{\frac{r+\delta}{\lambda_H}}] < 0$$

- Let $\Delta_L \to \Delta_H, r+\delta \to 0, \lambda_L$ large, then equality cannot be held $\Rightarrow$ PAM not an equilibrium

Conclusion

Economic implication

- Wages change faster in firms with faster learning
- Turnover is decreasing in tenure + different for experienced
- The wage could be increasing (H worker) or decreasing (L worker) in tenure
  - Relative to trend if there is human capital accumulation
- Can fully characterize wage distribution
- The variance of wage distribution is increasing in tenure

Conclusion

Theoretical Implication

- New no-deviation condition: from sequential rationality (holds trivially in standard bandit problem; from VM & SP)
- Show that uniqueness of cutoff equilibrium is restored
- SupM $\Rightarrow$ PAM even if signal-to-noise ratio dominates in L
- Robust to general Bayesian Learning

Assortative Learning

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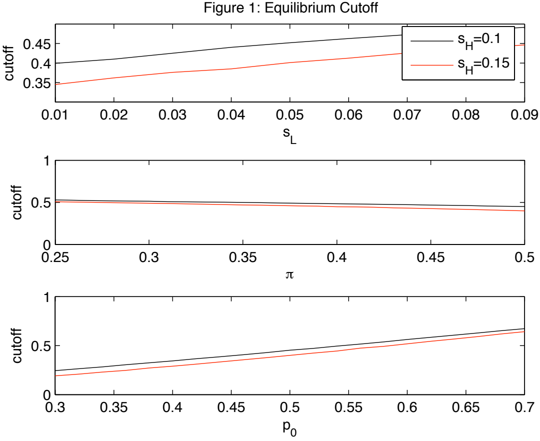
Equilibrium allocation and distribution

Comparative statics

Claim

$\underline{p}$ is strictly increasing in p_0 and decreasing in π.



Equilibrium allocation and distribution

Ergodic distribution

- Ergodic density f_y satisfies Kolmogorov forward equation

$$0 = \frac{df_y(p)}{dt} = \frac{d^2}{dp^2}[\Sigma_y(p) f_y(p)] - \delta f_y(p)$$

- with general solution:

$$f_y(p) = [f_{y0} p^{\gamma_{y1}} (1-p)^{\gamma_{y2}} + f_{y1} (1-p)^{\gamma_{y1}} p^{\gamma_{y2}}]$$

where

$$\gamma_{y1} = -\frac{3}{2} + \sqrt{\frac{1}{4} + \frac{2\delta}{s_y^2}} > -1 \text{ and } \gamma_{y2} = -\frac{3}{2} - \sqrt{\frac{1}{4} + \frac{2\delta}{s_y^2}} < -2.$$

Equilibrium allocation and distribution

Role of p_0

- The Kolmogorov forward equation is only valid for $p \neq p_0$ and there is a kink in the density function at $p = p_0$.
- There are two cases: $\underline{p} < p_0$ and $\underline{p} > p_0$.
- Note: entry from a non-degenerate distribution around p_0, but hard to solve differential equation explicitly

Equilibrium allocation and distribution

Equilibrium conditions

$$W_H(\underline{p}) = W_L(\underline{p}) \qquad \text{(Value Matching)}$$

$$W'_H(\underline{p}) = W'_L(\underline{p}) \qquad \text{(Smooth-pasting)}$$

$$W''_H(\underline{p}) = W''_L(\underline{p}) \qquad \text{(No-deviation)}$$

$$\Sigma_H(\underline{p}+)f_H(\underline{p}+) = \Sigma_L(\underline{p}-)f_L(\underline{p}-) \qquad \text{(Boundary condition)}$$

$$\int_{\underline{p}}^{1} f_H(p)dp = \pi \qquad \text{(Market clearing } H\text{)}$$

$$\int_{0}^{\underline{p}} f_L(p)dp = 1-\pi \qquad \text{(Market clearing } L\text{)}$$

$$\frac{d}{dp}[\Sigma_L(p)f_L(p)]|_{\underline{p}-} = \frac{d}{dp}[\Sigma_H(p)f_H(p)]|_{\underline{p}+} \qquad \text{(Flow equation at } \underline{p}\text{)}$$

$$f_H(p_0-) = f_H(p_0+) \qquad \text{(Cont. density at } p_0\text{)}$$

- 8 eq., 9 unknowns: $V_L, V_H, k_L, k_H, \underline{p}, f_{H0}, f_{H1}, f_{H2}, f_{L0}$ (indeterminacy of prices V_L as in Becker)

Equilibrium allocation and distribution

Existence and uniqueness

Theorem

Under strict supermodularity, for any pair $(p_0, \pi) \in (0,1)^2$*, there exists a unique PAM cutoff* $\underline{p}$*. Moreover,* $\underline{p} < p_0$ *if and only if:*

$$\left(\frac{p_0}{1-p_0}\right)^{\gamma_{H1}-\gamma_{L2}} \frac{\delta/s_H^2}{\delta/s_L^2} \frac{\int_{p_0}^{1} p^{\gamma_{H2}}(1-p)^{\gamma_{H1}}\,dp}{\int_0^{p_0} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}\,dp} < \frac{\pi}{1-\pi}.$$

Equilibrium Payoffs

- As in the frictionless case, there is indeterminacy in equilibrium payoffs.
- As usual, we assume $\mu_{LH} > \mu_{LL} = 0$ and then we can normalize $V_L = 0$.
- V_H is uniquely given by:

$$rV_H = (\mu_{LH} - \mu_{LL}) + \frac{\alpha_H(\alpha_L - 1)(\Delta_H - \Delta_L)\underline{p}}{\alpha_H(\alpha_L - 1) - (1 - \underline{p})(\alpha_L - \alpha_H)},$$

where

$$\alpha_y = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta)}{s_y^2}} \geq 1.$$

The Planner's Problem

Proof

1. Consider N cutoffs (generic. odd): $0 < \underline{p}_N < \cdots < \underline{p}_1 < 1$
2. Suppose $p \in (p_n, p_{n-1})$ match with L
3. move $(p_n, p_{n-1}) \rightarrow (p_n - \epsilon_2, p_{n-1} - \epsilon_1)$, s.t. ϵ_1, ϵ_2 satisfy market clearing
4. Only change f_L in $(\tilde{p}_n, \tilde{p}_{n-1})$ to $\tilde{f}_L$; keep all other f_H, f_L
5. Martingale property

$$\mathbb{E}_{\Omega_H} p + \mathbb{E}_{\Omega_L} p = \int_{\Omega_H} p f_H(p) dp + \int_{\Omega_L} p f_L(p) dp = p_0$$

6. Then $\mathbb{E}_{\Omega_H} p - \mathbb{E}_{\tilde{\Omega}_H} p > 0$ since by construction

$$\int_{p_{n-1}-\epsilon_1}^{p_{n-1}} f_H(p) dp = \int_{p_n-\epsilon_2}^{p_n} f_H(p) dp$$

7. Lemma: Higher $\mathbb{E}_{\Omega_H} p$ ($\Leftrightarrow$ lower $\mathbb{E}_{\Omega_L} p$) $\Rightarrow$ higher output