Indeterminacy

and

Sector-Specific Externalities

by

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and

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Abstract

We introduce, into a version of the Real Business Cycle model, mild increasing returns-to-scale. These increasing returns-to-scale occur as a consequence of sector specific externalities, that is externalities where the output of the consumption and investment sectors have external effects on the output of firms within their own sector. Keeping the production technologies for both sectors identical for expositional simplicity, we show that indeterminacy can easily occur for parameter values typically used in the real business cycle literature, and in contrast to some earlier literature on indeterminacies, for externalities mild enough so that labor demand curves are downward sloping. *Journal of Economic Literature* Classification Numbers: E00, E3, O40.

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1. Introduction

Recently there has been a renewed interest in "indeterminacy", or alternatively put, in the existence of a continuum of equilibria in dynamic economic models .[1] Part of the impetus for this renewed interest comes from the realization that indeterminacy can easily occur in real business cycle models or in models of endogenous growth that have been augmented to include elements of increasing returns, externalities or monopolistic competition, as in Baxter and King (1991) , Lucas (1988) or Romer (1990). An even more compelling reason that accounts for the renewed interest in these models, and in the possibility of indeterminacy, has been the empirical findings of Hall (1988), (1990), Caballero and Lyons (1992), Baxter and King (1991) and others, concerning the magnitude of externalities and of increasing returns which are critical for generating indeterminacies. The magnitudes of increasing returns, externalities or markups suggested by these studies can easily put the economy in the range of parameter regions that are consistent with indeterminacy.

In an earlier paper, Benhabib and Farmer (1994) showed that a necessary and sufficient condition for indeterminacy in a one sector growth model could be expressed in a relatively simple way. This condition required that externalities should be large enough to imply that the demand curve for labor should be upward sloping and further, that the slope of labor demand should exceed the slope of labor supply. Early estimates of externalities, for example, by Caballero and Lyons (1994) or Baxter and King (1991) found evidence of externalities that plausibly placed the economy within this range. But although the early estimates of externalities were relatively large, more recent estimates have called into question these results.[2] The purpose of this paper is to provide a version of a standard real business cycle model with sector specific rather than aggregate externalities that leads to indeterminacy for much smaller magnitudes of external effects than the earlier models, and for which the demand and supply curves for labor have the standard slopes.

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[1] See for example Benhabib and Farmer (1994), Benhabib and Perli (1994), Farmer and Guo (1994), ,Gali (1994a), (1994b), Xie(1994), Chamley(1993), Boldrin and Rustichini(1994), Beaudry and Devereux (1994), Schmitt-Grohe (1994), Cooper and Chatterjee (1994), Howitt and McAffee (1988) Benhabib, Perli and Xie (1995) as well as many others.

[2] See Basu and Fernald(1994a), (1994b), Norrbin (1993) and Bartlesman, Caballero and Lyons (1994) for new parameter estimates, or the comments of Aiyagari(1995) on Farmer and Guo (1995) on labor demand curves. See however the comments at the end of our section 8 about the new estimates.

We should note that it is only to keep our notation simple that we choose to stress externalities as a way of separating the competitive equilibrium of our model from the solution to a planners problem. However, as in our earlier work (see Benhabib and Farmer (1994)), there is an equivalent representation of our model in which increasing returns-to-scale are internalized by firms and in which a monopolistically competitive sector is used to provide a competitive theory of distribution.[3]

The intuition for the existence of indeterminacy in our model that is quite straightforward. Consider starting with an arbitrary equilibrium trajectory of investment or consumption, and inquire whether a faster rate of accumulation and growth can also be justified as an equilibrium. This would require a higher return on investment. If higher anticipated stocks of future capital raise the marginal product of capital by drawing labor out of leisure, or by reallocating labor across sectors, the expected higher rate of return may be self-fulfilling. Such a scenario will not work in a standard concave problem, since an increase in investment will increase the stock of capital and *lower* the rate of return, even when we account for the additional labor that may be drawn out of leisure and into production. If, on the other hand, there are sufficient increasing returns that are consistent with optimization, either because of externalities or because of imperfect competition that generate markups, these increasing returns may amplify the movement of labor into production and provide a sufficient boost to private rates of return to justify multiple equilibria. The critical parameters are the magnitudes of increasing returns or externalities, and the ease with which labor can be drawn into employment – that is – the elasticity of labor supply. (For an explicit treatment of this tradeoff see Figure 2 in section 8 below).

The intuition that we provide above will work in a one sector model. However, in this case the required magnitude of increasing returns or aggregative externalities that deliver indeterminacy may still be too large for reasonable values of the labor supply elasticity. By contrast, when we allow external effects in each sector to depend on the aggregate output of their own sector, factor reallocations across the sectors

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[3] See Benhabib and Farmer (1994). Cooper and Chatterjee (1994) provide a similar model in which intermediaries face fixed set up costs. The Cooper and Chaterjee (1994) model produces a Cobb-Douglas aggregate technology with increasing returns to scale when intermediate industries are also Cobb-Douglas with fixed costs. In their case an expansion of output over the business cycle produces an expansion in the number of intermediate industries. A modified form of the Cooper and Chaterjee technology with two sectors and sector specific intermediate producers leads to exactly the same social technology as our model with sector-specific externalities

can have strong effects on marginal products, and indeterminacies can occur with much smaller externalities than one requires in the one sector case.[4]

In sections 2-4 we describe the details of the model, beginning with the technology. The structure is similar to that of Benhabib-Farmer (1994) and like our earlier work it contains the standard real business cycle model as a special case. As in our earlier paper we will derive conditions for the steady state equilibrium of the economy to be indeterminate by formulating the model in continuous time – the continuous time results are cleaner than the discrete time dynamics and we are able, in the continuous time system, to find a simple necessary and sufficient condition for indeterminacy. Sections 2 and 3 describe the private and social technologies. Section 4 describes the preferences and the equilibrium. Section 5 focuses on dynamics and section 6 describes the steady state. Section 7 discusses local dynamics and how indeterminacy emerges. Section 8 provides an economic interpretation of the condition that generates indeterminacy and argues for its empirical plausibility. It also includes a discussion of the current estimates of increasing returns and external effects and their relevance for the results of this paper. Section 9 raises the issue of technology shocks and procyclical consumption in relation to the model, and explores some connections to the home production model of Benhabib, Rogerson and Wright (1991) and suggests avenues for future research on the topic of indeterminacy in a business cycle context. Finally there arises the question as to whether once an equilibrium selection device is imposed, the model's predictions are roughly in line with empirical observation. One possible method for equilibrium selection is to introduce sunspots. The appendix derives a discrete time version of the model and provides a calibration exercise that introduces sunspot shocks. It suggests that our model can generate time series that roughly approximate some of the main features of actual economic data.

2. The Private Technology

Unlike Benhabib and Farmer (1994), we assume that there are two distinct commodities that we refer to as an investment good, "I" and a consumption good "C". Each commodity is produced by a decentralized competitive sector that rents capital and labor in competitive factor markets. Letting "K" be the economy-wide stock of capital, "L" be the economy-wide stock of labor and μ_K and μ_L be the

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[4] For an empirical framework which assigns external effects to industries not through raw aggregate output but for outputs related to the immediate suppliers and customers of an industry, see Bartlesman Caballero and Lyons (1994).

fractions of K and L used in the consumption goods industry we can write the output of the two industries as follows:

(1) $$C = A(\mu_K K)^a (\mu_L L)^b,$$

(2) $$I = B(\{1-\mu_K\}K)^a (\{1-\mu_L\}L)^b,$$

where we impose the assumption of constant returns-to-scale in the technologies faced by individual firms, that is:

(3) $$a + b = 1.$$

Notice that the two industries use identical technologies with the exception of the two scaling factors "A" and "B" – we assume that from the perspective of the individual firms in the industry A and B are taken to be constant. From the perspective of the industry as whole, however, we allow A and B to depend on sector specific or economy wide use of capital and labor. We return to this point below.

The assumption of free entry into the two sectors implies that profits must be equal to zero in each industry. The first order conditions for profit maximization in each industry can be combined to find the relationship of μ_K and μ_L to relative prices and to the parameters of the technology. For the special case that we consider in this paper, the case for which factor intensities are identical across the two sectors, these conditions imply that:[5]

(4) $$\mu_K = \mu_L.$$

This result allows us to rewrite (1) and (2) in terms of the common factor share parameter that we refer to as μ –

(1') $$C = \mu A K^a L^b,$$

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[5]Letting q be the rental rate, w the wage in units of the consumption good and p be the price of the investment good the first order conditions for profit maximization in the two industries are given by:

(i) $\frac{aC}{\mu_K K} = q$, (ii) $\frac{bC}{\mu_L L}$ w, (iii) $\frac{apI}{(1-\mu_K)K} = q$, (iv) $\frac{bpI}{(1-\mu_L)L} = w$. Taking the ratios of (i) to (iii) and (ii) to (iv) it follows that $\mu_K = \mu_L$. Notice that this result relies on the assumption that factor intensities are the same in the two industries (the same parameters "a" and "b" appear in both technologies.)

(2')
$$I = (1-\mu)BK^aL^b,$$

and to find an expression for the production possibilities frontier, "ppf":

(5)
$$C + (A/B)I = C + pI = AK^aL^b \equiv Y$$

In equation (5) we denote aggregate output by Y and the relative price of the investment good by p For an economy with no externalities in which A and B are constant, the ppf is linear for given K and L and has slope p = (-A/B).

3. The Social Technology

Unlike the aggregate one sector model, in a two sector model externalities may be either aggregate or sector specific. The following specification allows for both possibilities:

(6)
$$A = \left(\bar{\mu}_K\bar{K}\right)^{a\theta}\left(\bar{\mu}_L\bar{L}\right)^{b\theta}\bar{K}^{a\sigma}\bar{L}^{b\gamma},$$

(7)
$$B = \left(\{1-\bar{\mu}_K\}\bar{K}\right)^{a\theta}\left(\{1-\bar{\mu}_L\}\bar{L}\right)^{b\theta}\bar{K}^{a\sigma}\bar{L}^{b\gamma}.$$

A bar over a variable denotes the economy-wide average and we assume that these economy-wide averages are taken as given by the individual firm. Thus, $\bar{\mu}_K\bar{K}$ is the average use of capital in the consumption goods industries and $\bar{K}$ is the economy-wide use of capital. The parameter θ represents a measure of sector specific externalities while the parameters σ and γ represent aggregate capital and labor external effects. We maintain the assumption throughout the paper that the two industries face the same sector specific externalities although this assumption could easily be relaxed. Using the result that competitive firms will choose to allocate capital and labor across industries in the same proportions, (equation 4), we can use equations (6) and (7) to write the *social technologies* in the consumption and investment industries as follows:

(8)
$$C = \mu^{1+\theta}K^{a(1+\theta+\sigma)}L^{b(1+\theta+\gamma)},$$

(9)
$$I = (1-\mu)^{1+\theta}K^{a(1+\theta+\sigma)}L^{b(1+\theta+\gamma)}.$$

To simplify notation we define the new parameters:

(10)
$$v \equiv (1+\theta),\ \ \alpha \equiv a(1+\theta+\sigma),\ \ \beta \equiv b(1+\theta+\gamma).$$

Using this notation we can find an expression for the *social production possibilities frontier*:

(11) $$C^{\frac{1}{v}} + I^{\frac{1}{v}} = K^{\frac{\alpha}{v}} L^{\frac{\beta}{v}} .$$

Note that v is greater than or equal to one; the case v = 1 corresponds to the absence of sector specific externalities. Similarly, α is greater than or equal to a and β is greater than or equal to b. The case of $\alpha/v = a$ and $\beta/v = b$ is the case of no aggregate externalities. By setting v = 1 the model collapses to the model with aggregate externalities that we studied in Benhabib-Farmer (1994) and for v = 1, α = a, and β = b, it collapses to the standard Cass-Koopmans model that forms the basis for the Real Business Cycle paradigm. Our main contribution in this paper is to show that for modest values of sectoral externalities, (values of v slightly greater than unity) the model displays indeterminacy and that a stochastic version of the model will therefore admit the possibility of business cycles that are driven by self-fulfilling beliefs.

Figure 1 illustrates that the existence of sectoral externalities implies that the social ppf will be concave. For example; suppose that the economy is at point P. The left panel of figure 1 illustrates the private opportunities of a competitive firm that contemplates transferring resources from the production of investment goods to the production of consumption goods. From the perspective of a single firm, holding constant the sectoral allocations of all other firms, the production possibilities frontier is linear with slope equal to (-A/B); the relative price of consumption and investment goods. The right-panel, on the other hand, illustrates the social opportunities of transferring resources from the investment sector to the consumption sector if this transfer is accomplished by all firms at the same time. To the social planner, the opportunity set is concave since the presence of sectoral specific externalities causes agglomeration effects in each sector. The curve in the right hand panel of figure 1 represents the production possibilities set of

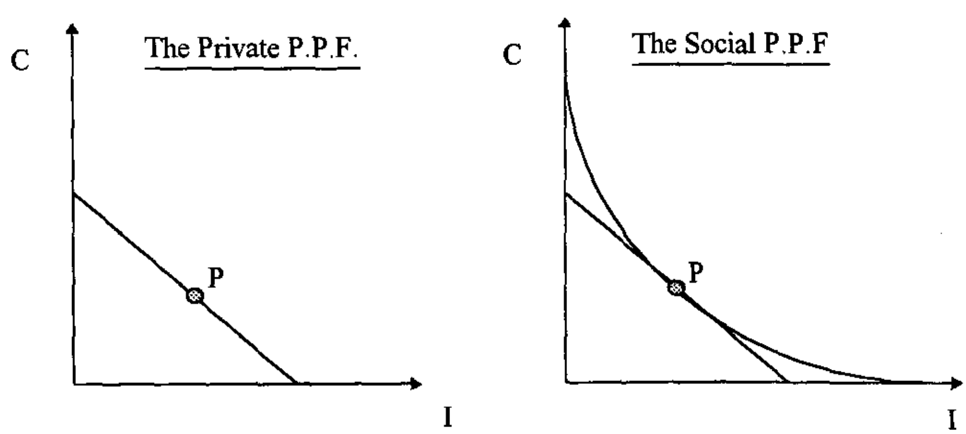

Figure 1: The Private and Social Production Possibilities Frontiers Compared

society. Superimposed on this same figure is the linear production possibilities frontier as perceived by an individual firm; notice that this private ppf is tangent to the social ppf at point P.

4. Preferences and the Solution to the Individual's Problem

We describe the preferences of a representative family in our economy by the period utility function:

$$U(C,L) = \ln C - \frac{L^{1+\chi}}{1+\chi}. \tag{12}$$

The representative family is assumed to choose L and C to maximize the discounted present value of utility using discount parameter ρ subject to the perceived production possibilities set:

$$C = AK^{a}L^{b} - (A/B)I, \tag{13}$$

and the law of motion for capital accumulation:

$$\dot{K} = I - \delta K, \tag{14}$$

where δ is the depreciation rate. Substituting (13) into (12) we can write the Hamiltonian for this problem as:

$$H = \ln\left[AK^{a}L^{b} - (A/B)I\right] - \frac{L^{1+\chi}}{1+\chi} + \Lambda(I - \delta K) \tag{15}$$

The first order conditions for maximization lead to the equations:

$$\frac{1}{C}b\frac{AK^{a}L^{b}}{L} = L^{\chi}, \tag{16}$$

$$\left(\frac{A}{B}\right)\frac{1}{C} = \Lambda, \tag{17}$$

together with the equation of motion for the co-state variable, Λ:

$$\dot{\Lambda} = (\rho + \delta)\Lambda - \left(\frac{1}{C}a\frac{AK^{a}L^{b}}{K}\right). \tag{18}$$

These three equations, together with the equation of motion of the capital stock;

$$\dot{K} = I - \delta K, \tag{19}$$

the definition of the private transformation function:

(20) $$C = AK^a L^b - (A / B)I,$$

and the transversality condition

(21) $$\lim_{t\to\infty} e^{-\rho t}\Lambda K = 0$$

completely describe the solution of a representative family for given values of A and B.

5. The Dynamics of Market Equilibrium

In this section we are going to impose the assumption that the average aggregate stocks of capital and labor, $\bar{K}$ and $\bar{L}$ and the average aggregate allocation of resources between sectors, $\bar{\mu}$ are each equal to the individual values of these variables, K L and μ. In words, each individual family acts in isolation taking the actions of other families as given but, in a symmetric equilibrium, every family takes the same actions. To study the dynamics of a competitive equilibrium we solve for the external parameters A and B and for the aggregate sectoral allocation μ in terms of the variables, K, L, C and Λ. By substituting these functions into the solutions for the individual optimizing problem we are able to analyze the dynamics of a competitive equilibrium.

We start with a definition. From equation (8) we define the new variable "S":

(22) $$S \equiv \frac{1}{\mu} = \frac{K^{\frac{\alpha}{v}} L^{\frac{\beta}{v}}}{C^{\frac{1}{v}}}.$$

"S" is the inverse of the factor share going to the consumption sector and it takes values between one and infinity. When S equals one all of the resources of society are allocated to consumption; when S equals infinity all of society's resources are allocated to investment. "S" is a key variable in determining the dynamics of a competitive equilibrium. Using the definition of S and the definitions of the externality parameters A and B (equations 6 and 7) we can rewrite A and the ratio of A to B in terms of S, K and L:

(23) $$A = \frac{K^{\alpha-a} L^{\beta-b}}{S^{v-1}},$$

(24) $$\left(\frac{A}{B}\right) = \frac{1}{(S-1)^{v-1}}.$$

Notice that when β equals b, α equal a, and v equals 1 both of these terms reduce to one. This is the case of no externalities. The term A is the externality in the consumption industry, B is the externality in the investment goods industry and (A/B) is the relative price of consumption goods to investment goods. Using these definitions of A and B we can rewrite the static first order equations from the agent's problem together with the definition of the social ppf:

(25) $$bS = L^{1-\chi},$$

(26) $$C(S-1)^{v-1} = \frac{1}{A},$$

(27) $$I = C(S-1)^{v}.$$

Equations (25) (26) and (27) are equivalent representations of the two first order conditions (16) and (17) and the ppf (20) that use the assumption of symmetric equilibrium, the definition of S from equation (22) and the expressions for A and (A/B) (equations 23 and 24).

Although we are treating the problem "as if" employment in each sector is allocated by a representative family, one might also think of decentralizing the labor market into households and firms. Using the decentralized conditions, representative firms in the consumption and investment sectors would equate the marginal product of labor to the real wage. Using the symbol w to represent the wage measured in units of the consumption good we can write the first order conditions for the household and for a firm in the consumption sector as follows:

(28) $$\left(\frac{bC}{\mu L}\right) = \left((\mu K)^{\alpha}(\mu L)^{\beta-1}\right) = w = L^{\chi}C.$$

The expressions on the left of this equation represents the marginal product of labor in the consumption sector. The right hand side is the ratio of the marginal utility of leisure to the marginal utility of consumption so that (28) is the labor market equation for the consumption good. To get the appropriate condition for the investment sector, first note that if we divide (1') by (2') and rearrange we obtain:

(29) $$pI\left(\frac{\mu}{1-\mu}\right) = C$$

where p = A/B, represents the relative price of the investment good. Combining (28) with (29) we have

(30) $$\left(\frac{pbI}{(1-\mu)L}\right) = pb\left(\left((1-\mu)K\right)^{\alpha}\left((1-\mu)L^{\beta-1}\right)\right) = w = L^{\chi}C$$

which is the labor market equation for the investment good. We will use (28) and (30) to interpret our results on indeterminacy further below.

In our discussion of the first order conditions we introduced a new variable, S that represents the inverse of the fraction of resources allocated to consumption. Our strategy for analyzing the properties of the equilibria of this model is to find a pair of dynamic equations in the state variable "K" and the co-state variable Λ and to analyze the properties of these equations in the neighborhood of a stationary state. The advantage of introducing the variable "S" follows from the fact that the dynamics of the system in the two variables Λ and K has a particularly simple representation:

(31) $$\frac{\dot{\Lambda}}{\Lambda} = \rho + \delta - a\frac{S}{\Lambda K},$$

(32) $$\frac{\dot{K}}{K} = \frac{S-1}{\Lambda K} - \delta.$$

In the next section we analyze these equations to find how the system behaves in the neighborhood of a stationary equilibrium by finding an expression for S in terms of the variables Λ and K.

6. Analyzing the Stationary State

Setting $\dot{\Lambda}$ and $\dot{K}$ equal to zero it follows that the steady state values of S and of ΛK are given by the expressions:

(33) $$\tilde{S} = \frac{\rho+\delta}{\rho+\delta(1-a)},$$

(34) $$\widetilde{\Lambda K} = \frac{a}{\rho+\delta(1-a)},$$

where a tilde over a variable denotes its value at the steady state. Using equations (22), (25) and (26) we can also solve for the steady state values of L, K, C and Λ.

(35) $$\tilde{L} = \left(b\tilde{S}\right)^{\frac{1}{1+\chi}},$$

(36) $$\tilde{K} = \left[\frac{\tilde{L}^{-\beta}\tilde{S}^{v}\delta^{1-v}\left(\rho+\delta(1-a)\right)^{v}}{a^{v}}\right]^{\frac{1}{\alpha-1}},$$

(37) $$\tilde{C} = \frac{\delta^{1-v}\left(\rho+\delta(1-a)\right)^{v}}{a^{v}}\tilde{K},$$

and

(38) $$\tilde{\Lambda} = \frac{a}{\rho+\delta(1-a)}\frac{1}{\tilde{K}}.$$

Notice from equation (36) that the system only has a stationary state in K if $\alpha \neq 1$. The case when $\alpha = 1$ leads to endogenous growth and we exclude this case in the current paper although it is an interesting model in its own right.

7. Local Dynamics

In this section of the paper we analyze the local dynamics of equations (31) and (32) around the stationary state $\tilde{\Lambda}$ and $\tilde{K}$. The analysis is simpler if we transform the equations by taking logarithms of all of the variables. Using lower-case letters to represent logs we can write the two dynamic equations in the form:

(39) $$\dot{\lambda} = \rho + \delta - ae^{s-k-\lambda},$$

(40) $$\dot{k} = e^{s-k-\lambda} - e^{-\lambda-k} - \delta.$$

As long as $v - \frac{\beta}{(1+\chi)} + (1-v)\frac{\tilde{S}}{(\tilde{S}-1)} \neq 0$ the implicit function theorem allows us to use equation (22) (the definition of S), and equations (25) and (26) (the two static first order conditions) to write c, k and s as functions of λ and k. For the case of the variable s the required function is implicitly defined by the equation:

(41) $$(S-1)^{1-v}S^{v-\frac{\beta}{1+\chi}} = b^{\frac{\beta}{1+\chi}}K^{\alpha}\lambda,$$

which also can be used to find the logarithmic derivatives of the required function s(λ, k). These partial derivatives are defined as follows:

(42) $$\frac{\partial s}{\partial \lambda} \equiv s_\lambda = \left\{ \frac{1}{v - \frac{\beta}{(1+\chi)} + (1-v)\frac{\tilde{S}}{(\tilde{S}-1)}} \right\},$$

(43) $$\frac{\partial s}{\partial k} \equiv s_k = \left\{ \frac{\alpha}{v - \frac{\beta}{(1+\chi)} + (1-v)\frac{\tilde{S}}{(\tilde{S}-1)}} \right\}.$$

Notice that $s_k = \alpha s_\lambda$. The elasticity of s with respect to λ evaluated at the steady state is a key parameter in our analysis since it turns out that the sign of s_λ holds the key to indeterminacy in this model. We show below that $s_\lambda < 0$ is a necessary condition for the steady state to be indeterminate.

The Jacobian of the system of equations (39) and (40), evaluated at the steady state, is given by the matrix:

(44) $$J = \begin{bmatrix} -(\rho+\delta)(s_\lambda - 1) & -(\rho+\delta)(s_k - 1) \\ \frac{(\rho+\delta)(s_\lambda - 1)}{a} + \frac{(\rho+\delta)}{a}\left(1 - \frac{\delta a}{\rho+\delta}\right) & \frac{(\rho+\delta)(s_k - 1)}{a} + \frac{(\rho+\delta)}{a}\left(1 - \frac{\delta a}{\rho+\delta}\right) \end{bmatrix}$$

which has a trace and a determinant given by expressions (45) and (46):

(45) $$TR = \frac{(\rho+\delta)}{a}\left((\alpha - a)s_\lambda + \frac{\rho a}{\rho+\delta}\right),$$

(46) $$DET = \frac{(\rho+\delta)^2}{a}\left(1 - \frac{\delta a}{\rho+\delta}\right)(\alpha - 1)s_\lambda.$$

The trace of the Jacobian is equal to the sum of the roots and the determinant is the product of the roots of the dynamical system (39) – (40) evaluated at the steady state. Since the system has one predetermined variable, K, and one non-predetermined variable, Λ, local indeterminacy requires that both roots of the system should be negative evaluated at the steady state. An equivalent condition is that the trace of the Jacobian should be negative and the determinant should be positive. Since we are considering models with relatively modest externalities the parameter α will be less than one and it follows from equation (46) that a necessary condition for a positive determinant is that:

(47) $$s_\lambda < 0.$$

The condition that the determinant is positive guarantees only that both roots have the same sign. Necessary and sufficient conditions for indeterminacy also require that the trace be negative; note that negative s_λ is not enough to guarantee that both roots are negative since the trace also contains a positive term, the magnitude of which depends on rate of time preference ρ. In practice, indeterminacy occurs in parameterized systems for relatively mild values of externalities. For versions of the model with no externalities, one can show that s_λ is positive. As sectoral externalities increase from zero a bifurcation occurs that changes the sign of s_λ, however, the bifurcation occurs as s_λ passes through plus infinity to minus infinity rather than moving through zero. Because of this route to indeterminacy the sufficiency condition for indeterminacy is easily satisfied close to the bifurcation point at which s_λ switches sign. Increasing externalities further or decreasing the inverse of the labor elasticity parameter χ can cause the trace to change sign again while the determinant remains positive. This indicates that two complex roots have their real parts change sign as the trace crosses from negative to positive; a classic Hopf bifurcation which indicates the presence of cycles. If cycles occur for the parameter region for which the trace is positive, they may be attracting and surrounding a completely unstable steady state. In this case we would still have indeterminacy since arbitrary choices of k and λ in the neighborhood of the cycle would lead the equilibrium trajectories to converge to the cycle and satisfy transversality conditions. Since this type of indeterminacy may involve larger and maybe unrealistic externalities or overly elastic labor supply, in this paper we will concentrate on indeterminacies that are associated with parameter regions where the steady state trace is negative. (See also Figure 2 below.)

8. Interpreting the Condition for Indeterminacy

Our earlier work (Benhabib and Farmer (1994)) is a special case of the model that we are studying here in which there are no sectoral externalities; for this case the parameter v is equal to one. In equation (48) we substitute the steady state values for $\tilde{S}$ into the expression for s_λ given by (42). It is clear from this

(48) $$\frac{\partial s}{\partial \lambda} \equiv s_\lambda = \left\{ \frac{1}{v - \frac{\beta}{(1+\chi)} + (1-v)\frac{\rho+\delta}{(\delta a)}} \right\}$$

equation that, when v is equal to one, the condition for s_λ to change sign is equivalent to the statement that $\beta-1$ should exceed χ. This is equivalent to the condition derived in Benhabib and Farmer (1994) that the demand for labor should slope up more steeply than the supply of labor.

In our earlier work we interpreted the condition for indeterminacy in terms of the slopes of the demand and supply curves for labor in a one sector model. We can find a similar condition in the model with sectoral externalities although there are now two labor demand curves – a demand-for-labor in the consumption sector and a demand-for-labor in the investment sector. To derive a log-linear form of these two demand functions we may take logs of equations (28) and (30):

$$\chi\ell n(L)+\ell n(C)=\ell n(b)+\alpha\ell n(\mu K)+(\beta-1)\ell n(\mu L) \tag{49}$$

$$\chi\ell n(L)+\ell n(C)=\ell n(p)+\ell n(b)+\alpha\ell n((1-\mu)K)+(\beta-1)\ell n((1-\mu)L) \tag{50}$$

In each of these equations, the left-side of the equation represents the supply curve of labor, holding constant consumption. This expression would be equated, by a representative household, to the logarithm of the real wage. The right-side of equations (49) and (50) represent the demands-for-labor in the consumption and investment sectors; holding constant the sectoral use of capital and the relative price of investment goods. It is clear from these equations that the slopes of the demand curves for the logarithm of labor in both sectors is β-1 and the slope of the supply curve for the logarithm of labor is χ. When sector specific externalities are present the condition for indeterminacy, that s_λ be negative, does not require that the labor demand curve in either sector should be upward sloping or have a slope greater than that of the labor supply curve.

For comparison with the econometric results obtained in one-sector models we may also obtain an aggregate labor demand curve that includes the effects of relative price changes. Putting together equations (28) and (30) we can sum labor demands in each sector to arrive at the aggregate labor demand curve:

$$b\frac{(pI+C)}{L} = w. \tag{51}$$

But since p=A/B we can replace (pI + C) in equation (51) by AK^aL^b using the definition of the ppf (equation 5). Finally, from the equilibrium value of A (equation 23) we can write the aggregate labor demand curve as:

(52) $$b\frac{K^{\alpha}L^{\beta-1}}{S^{v-1}} = w.$$

It is clear from equation (52) that the position of the aggregate labor demand curve depends not only on the aggregate stock of capital, but also on the allocation of resources across sectors, (the variable S). Suppose that an econometrician were to mis-specify the model assuming incorrectly that the economy has one sector and hence missing the effect of S from the demand function. We could interpret his results in terms of the sectoral externality model by finding a reduced form labor demand function that eliminates the effect of S using the fact that $L^{1+\chi} = bS$ from equation (25). Using this result and taking logs of (52) we can describe the following aggregate labor market equation,

(53) $$\chi\ell n(L) + \ell n(C) = \text{constant} + \alpha\ell n(K) + (\beta - 1 - (1+\chi)(v-1))\ell n(L).$$

The right-side of this equation represents the economy wide labor demand curve that would be estimated by an economist who mistakenly specified the economy as a one-sector model, ignoring the effects of sectoral externalities. Note that the labor demand curve in this mis-specified economy would be downward sloping if $(\beta - 1 - (1+\chi)(v-1)) < 0$, a condition that is easily satisfied if $\beta - 1 < 0$.

In fact it is surprisingly easy to obtain indeterminacy with downward sloping labor demand and upward sloping supply curves and with parameter values that are typically used in the real business cycle literature. The most important feature of the indeterminacy condition in the sectoral specific model is that indeterminacy is consistent with very small values of sectoral externalities and with demand curves that slope down and supply curves for labor that slope up.[6] Suppose for example that there are *no aggregate externalities* implying that β/v is equal to b and α/v equals a. A set of parameter values, typically used in the real business cycle literature, that are consistent with indeterminacy are given below, together with the steady state values that they imply for the endogenous variables, L, pK/Y, C/Y and pI/Y where Y = C+pI:

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[6] In the calibration literature it is common to assume logarithmic preferences over consumption. For the standard specification of utility that we use above, the steady state value of the parameter that plays the same role as χ is given by the ratio of time spent working to time spent in leisure – a value that is often calibrated at around 1/4 – implying a labor supply elasticity of 4. For this value of χ the supply curve slopes up with slope 1/4. We choose a more conservative value of $\chi = 1$ that makes indeterminacy harder to obtain. See Figure 2 below.

Parameter	*Calibrated Value*
b	0.7
a	0.3
v	1.15
ρ	0.05
δ	0.1
χ	1

Variable	*Steady State Value*
L	0.935
pK/Y	2.00
C/Y	0.80
pI/Y	0.20

For the above parameter values $\beta-1 < 0$ so that labor demand is downward sloping and other parameters are well within the range that is common in the literature. We can illustrate the region of indeterminacy associated with parameters for the inverse labor elasticity χ and the externality parameter θ (where $\theta = v-1$), keeping the other parameters unchanged. The shaded region in Figure 2 represents the region of indeterminacy in the χ-θ space. Note that the lower the values of χ, the easier it is to get indeterminacy. (Note also that the region where $\chi > 0$ and below the lower curve where the trace is positive can also represent a region of indeterminacy with a totally unstable steady state but an attracting cycle, as discussed at the end of section 7 above.)

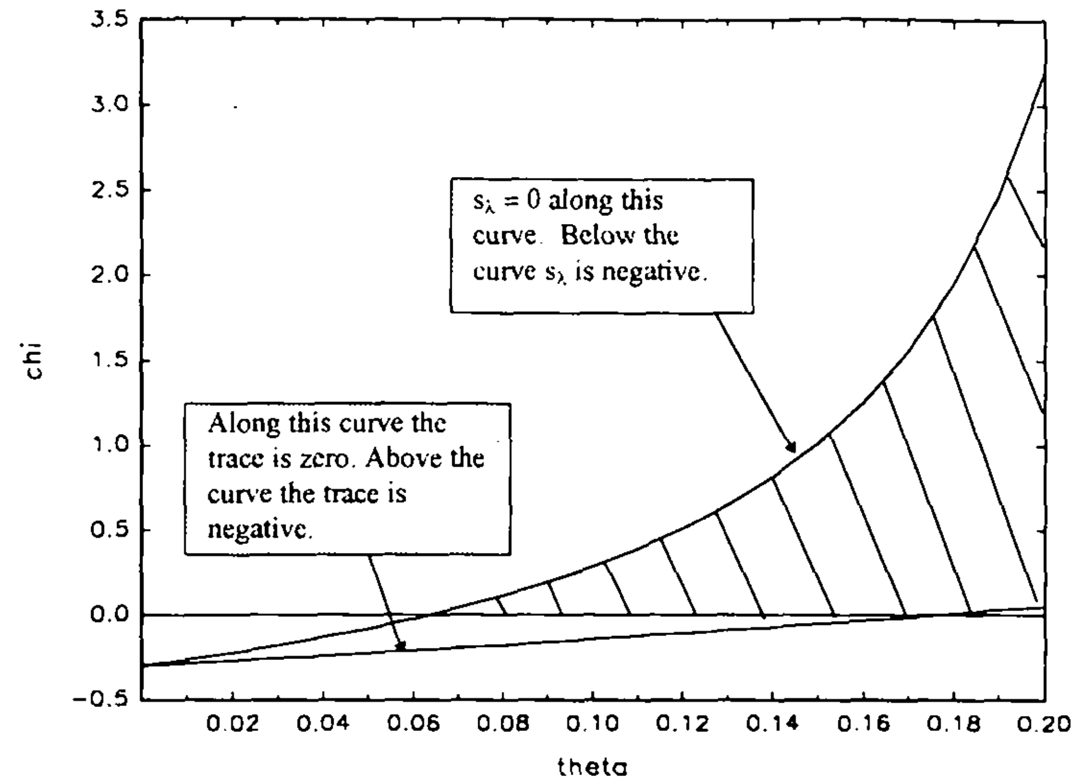

FIGURE 2

Figure 2 indicates that indeterminacy can be obtained with the externality parameter θ that is as low as 0.064 and with all the other parameters well within acceptable ranges. Earlier estimates of Hall (1988), (1990), Domowitz et al. (1988), Caballero and Lyons (1992) or Baxter and King (1990) suggest that the elasticity of aggregated output with respect to inputs should be higher than that suggested by factor shares, often by a factor of 40-60%[7]. More recent work by Basu and Fernald (1994a), (1994b) is critical of the earlier methodologies that estimate external effects and increasing returns because they seem to ignore the share of intermediate goods in computing the Solow residual and its correlation with output aggregates. They mostly argue that returns to scale are approximately constant and that markups are

[7] In Benhabib and Farmer (1994) in discussing the monopolistically competitive case we assume that there are no excess profits and no fixed costs. This implies that the markup will be equal to the degree of increasing returns. The increasing returns estimates of Basu and Fernald (1994b) cited below are obtained after adjusting for positive profit rates of 5%, which are likely to be high for reasons cited in their paper.

small. Their best estimate of the degree of increasing returns corresponds to a value of our parameter $v = 1.03$, (v is equal to $1+\theta$). Similar estimates by Morrison (1990) that does account for the usage of intermediate goods yield a higher estimate of $v = 1.12$. Norrbin (1993) examines 21 manufacturing industries. His methodology includes intermediate inputs and he finds markups to be smaller than the earlier estimates of Hall (1990). His average estimates for markups are 14%-18%, depending on whether markups or their inverses are estimated. More recently Bartlesman, Caballero and Lyons (1994), using gross output data which also does not exclude shares of intermediate goods, find that external effects associated with aggregate output measures weighted to reflect the immediate suppliers or customers of the industry, to be around 1.12 in the short-run and around 1.30 over the longer horizon. Furthermore, as Basu and Fernald (1994b) also note, intermediate goods themselves will also be produced with markups or with externalities and under increasing returns, so that the elasticity of aggregated outputs like consumption or investment with respect to capital and labor inputs will have to be higher than the estimates that are based on disaggregated outputs. Thus it is quite possible that as external effects and markups implicit in intermediate goods pile up in aggregation, the magnitude of increasing returns for the aggregated sectoral outputs will be closer to the higher estimates obtained say by Baxter and King (1991).[8] In any case, our point is that the degree of increasing returns required to generate indeterminacy in our model calibrated to standard business cycle parameters is quite low, somewhere in the order of 1.10 to 1.15. These magnitudes are likely to be even lower if we were to further disaggregate the theoretical model with sector specific externalities. It seems therefore that even the lower estimates of increasing returns (or decreasing costs that must be present with some fixed costs) are quite sufficient to make an empirically plausible case for the indeterminacy of equilibrium in our simple model.

9. Indeterminacy and Procyclical Consumption

One feature that deserves discussion is the fact that, without technology shocks and with small externalities, our model predicts that investment and employment will be procyclical but that consumption will be countercyclical. Since we do not explicitly model shocks, we can take

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[8] This point was communicated to us by Michael Woodford. We may consider a case for example where the aggregate or some of the sectoral outputs are produced with intermediate goods so that $Y = I^a I^e$ where I^e represents an external effect. Similarly suppose intermediate goods are produced with labor alone: $I = L^b L^e$. While the measure of externalities in each sector is e, for the aggregate economy $Y = (L^{b+e})^{a+e}$ and the aggregate externality is $(a+b)e + e^2$, which is greater than e if $a+b>1$.

countercyclical consumption to mean that consumption and output will move in opposite directions, either as the economy moves along an equilibrium path where we ignore changes in the capital stock for the short run, or if the economy jumps to another equilibrium path. Making use of equations (1), (22), (25) and (29) we can derive the following three equations to illustrate this idea:

(54) $$C + pI = \mu^{\theta} K^{a} L^{b} = S^{1-v} K^{\alpha} L^{\beta} = b^{v-1} K^{\alpha} L^{\beta-(1+\chi)(v-1)}$$

(55) $$C = b^{v} K^{\alpha} L^{\beta - v(1+\chi)}$$

(56) $$I = K^{\alpha} L^{\beta} \left(1 - bL^{(1+\chi)}\right)^{v}$$

It is clear from (54) that output, $C + pI$, (the measure of GDP in this economy) will be positively related to employment, L, if $\beta - (1+\chi)(v-1) > 0$, which is likely to be the case for reasonable parameterizations of the externality and the labor supply elasticity. It also follows from equation (56) that employment will be positively correlated with investment. From equation (55), however, it follows that consumption will be negatively related to employment unless the externality is large, that is if $\beta - (1+\chi)v > 0$. This reflects the familiar result from the real business cycle literature, that since capital moves little in the short-run, consumption tends to be countercyclical in a neoclassical model *without* technology shocks.

A closer look may help clarify some theoretical approaches and empirical issues that are relevant for our paper. Let $U'(C)$ be the marginal utility of consumption, $V'(-L)$ be the marginal utility of leisure and MPL(L) the marginal product of labor. The first order condition for the choice of labor in a standard one-sector model takes the form:

(57) $$U'(C)MPL(L) = V'(-L)$$

Suppose that employment increases spontaneously in this model, as would be the case if "sunspots" were the dominant source of f luctuations. In this case the increase in L would decrease MPL and increase V' and equality will be restored only if C falls and U' rises. In other words, sunspot fluctuations will cause consumption to be countercyclical. In the following discussion we identify three channels that might break this link.

(1) The first possibility is that demand and or supply curves may have non-standard slopes. If the marginal product of labor, MPL, is increasing in L, which gives an upward sloping labor demand, or if V' is decreasing in L, which gives a downward sloping labor supply then an increase in L may be associated with an *increase* in consumption and equation (57) could still hold. When we estimate a model that involves (57), the procyclical consumption in the data may well force the estimated parameters to imply an upward sloping demand, a downward sloping supply, or both – this, for example, is exactly what Farmer and Guo (1994) find when they estimate a one sector model. The existence of an upward sloping demand curve for labor requires externalities or monopolistic competition, but a downward sloping supply curve can occur even when utility functions are concave. For example, an alternative specification of utility that permits pro-cyclical consumption would replace $U'(C)$ and $V'(-L)$ with $U_1(C,L)$ and $U_2(C,L)$. This non-separability may allow the labor supply curve to slope down even in the absence of externalities. However, one may show that a downward sloping labor supply curve also implies that consumption is an inferior good.[9] Since we find it implausible that a representative household that won the lottery would *decrease* its consumption this route to procyclical consumption does not seem to be fruitful, at least when consumption and leisure are the only two commodities.

(2) A second way in which one may reintroduce procyclical consumption follows from work on monopolistic competition. In this setting the relevant variable for equation (57) is not MPL but MPL adjusted for the markup. If the markup is constant the conclusions that follow from (57) are unchanged, but if the markup is countercyclical, then procyclical consumption can be rescued, as is the case in Rotemberg and Woodford (1991), (1992) and for different theoretical reasons in Gali (1994a).

(3) All of the above discussion is concerned with the difficulty of explaining procyclical consumption in a model in which all shocks arise form sunspots as in Farmer and Guo (1994), for example. Procyclical consumption should be easier to obtain with technology shocks since in this case output may rise sufficiently to allow both investment and consumption to increase in response to a positive shock, even though labor may move out of the production of consumption goods to the production of investment

[9] We thank Michael Woodford and Stephanie Schmitt-Grohé for (independently) pointing this out to us in private communications. In fact Mankiw, Rotemberg and Summers (1985) using wage data also estimate the analog of (57) with a flexible utility function and find that either leisure or consumption must be inferior even with technology shocks, since such shocks should be reflected in the wage data.

goods. Indeterminacy would still remain, so that given the capital stock and the realization of the technology shock, investment and consumption would not be uniquely determined. In other words, even if one thinks that technology shocks provide the impulse to the business cycle – indeterminacy still has a considerable amount to add to the story by providing a plausible explanation of an endogenous propagation mechanism. Our model, driven by technology shocks, could conceivably provide a convincing explanation of the autocorrelation properties of business cycle data *even when driven by i.i.d. shocks.* A related approach which we pursue in the calibrated discrete time model in the appendix explores the possibility of sunspot shocks correlated with the technology shocks. This structure may capture the idea that sunspots are simply overreactions to news about fundamentals and also serves to bolster the correlation between output and consumption.

Although technology shocks are probably important in practice, the real business cycle approach with technology shocks alone still does not resolve the issue of procyclicality completely, since, *employment* in the consumption sector must remain countercyclical and this is not consistent with data. A more promising approach is to introduce a naturally countercyclical sector that will feed labor into the economy during booms and absorb labor during recessions. The "home" sector, as shown by Benhabib Rogerson and Wright (1991) will serve that purpose, even in the absence of technology shocks, and will deliver procyclical consumption as well as procyclical employment in the consumption sector. In such a setup ignoring the home sector and the movements of labor between home and market may indeed make it seem as if leisure is inferior (see footnote 10). Some preliminary work already incorporating home production into a model with indeterminacy has been undertaken by Perli (1994). A related approach would be to introduce either a "search" or a "school-human capital" sector into the model, which may create a countercyclical sector that absorbs labor. We hope to pursue this approach in future work..

10. Conclusion

The idea that "indeterminacy" may provide a plausible explanation of the propagation mechanism in U.S. business cycles has recently received a considerable degree of attention in the literature. In spite of the wide attention that the topic has received however, there is still resistance to the idea of indeterminacy based, in part, on the fact that existing models seem to require an unreasonably high degree of increasing returns-to-scale. Our intent, in this paper, has been to show that a relatively mild move away from the one-sector model allows for indeterminacy in calibrated models of business cycles with much more reasonable degrees of externalities or increasing-returns-to-scale than those required in earlier work.

We have shown, in particular, that the large external effects that gave rise to upward sloping demand curves for labor in previous works are not required to generate indeterminate equilibria and that the two-sector model allows for indeterminacy with downward sloping labor demand curves and upward sloping labor supply curves when the values of externalities are within even the strictest of recent estimates at the industry level. Our personal interpretation of this work is that indeterminacy is an empirically plausible phenomenon that requires further careful scrutiny. We think that the payoffs from this strategy are high since, by pursuing empirical models with potentially indeterminate equilibria, it becomes possible to find a convincing *endogenous* explanation for the propagation mechanism in U.S. business cycles. By following the econometric strategies outlined in Farmer and Guo (1995) one might hope to use models in this class both to forecast and provide a guide to policy analysis.

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Appendix A

In this appendix we derive a stochastic discrete time version of our model and calibrate it along the lines of real business cycle models. While analytic characterizations of indeterminacy would be more complex in discrete time, it is easy to check for it in particular parametrized examples. Not surprisingly, indeterminacy obtains in the discrete time version of our model for reasonable and standard parametrizations of an economy with externalities. We introduce sunspot and technology shocks into our model and calibrate it in the standard manner of real business cycle analysis. The results of the calibration are given and discussed at the end of the appendix.

Step 1: Defining the Household Problem

The first step is to define the maximization problem solved by a representative family in the economy. We assume that each family allocates time between leisure and work and that it may use time allocated to work to produce either consumption or investment goods. The private technology in each industry is given by equations (A1) and (A2):

(A1) $$C_t = A_t U_t \mu_t K_t^a L_t^b,$$

(A2) $$I_t = B_t U_t (1 - \mu_t) K_t^a L_t^b,$$

where μ_t is the fraction of resources allocated by the family to consumption – we show in the paper that – given the asumption that the parameters a and b are the same in each industry – the family will choose to allocate the same fraction of capital and labor to each sector. An equivalent way of stating the technological constraints is in the form of the private ppf, defined by the equation:

(A3) $$C_t = U_t A_t K_t^b L_t^b - \frac{A_t}{B_t} I_t.$$

The random variable U_t represents an economy wide productivity shock. We assume that U_t has an unconditional mean of U which we take to be a constant term in the production function. The capital accumulation identity is given by:

(A4) $$K_{t+1} = K_t (1 - \delta) + I_t.$$

Replacing I in (A3) from (A4) gives us a constraint on the individual's maximization problem that must hold for each value of t.

(A5) $$C_t = U_t A_t K_t^b L_t^b - \frac{A_t}{B_t}\left(K_{t+1} - (1-\delta)K_t\right.$$

Each household in the economy chooses a sequence of values of L_t and K_{t+1} to maximize the following utility function:

(A6) $$\text{Max } E_t\left[\sum_{t=1}^{\infty} \frac{1}{(1+\rho)^t}\left\{\log(C_t) - \frac{L_t^{1+\chi}}{1+\chi}\right\}\right]$$

subject to the sequence of constraints defined by (A5) and taking A_t *and* B_t *as given.*

Step 2: Endogenously Determining the Externality Parameters

As in the paper we define economy wide externalities:

(A7) $$A_t = \mu_t^{v-1} K_t^{\alpha-a} L_t^{\beta-b},$$

(A8) $$\frac{A_t}{B_t} = \left(\frac{\mu_t}{1-\mu_t}\right)^{v-1},$$

where μ is the average aggregate share of resources devoted to consumption. We impose the assumption that all households are identical and we search for the values of A and B in a symmetric equilibrium. Replacing (A7) in equation (A1) (the private production technology for consumption) we arrive at a definition of the variable S:

(A9) $$S_t = \frac{K_t^{\alpha/v} L_t^{\beta/v} U_t^{1/v}}{C_t^{1/v}} = \frac{1}{\mu_t}.$$

Replacing the definition of S in (A7) and (A8), we can express the value of the externality parameters A and B in terms of K,L and S.

(A10) $$A_t = \frac{K_t^{\alpha-a} L_t^{\beta-b}}{S_t^{v-1}},$$

(A11) $$\frac{A_t}{B_t} = \frac{1}{(S_t - 1)^{v-1}}.$$

The term A/B is the slope of the social ppf – notice that this depends on the fraction of economy wide resources allocated to consumption (the inverse of S).

Step 3: The Household Decision Rules

Maximizing (A6) subject to (A5) leads to the following two first order conditions:

(A12) $$L_t^{\chi} C_t = b U_t A_t K_t^a L_t^{b-1},$$

(A13) $$\frac{1}{C_t}\frac{A_t}{B_t} = E_t\left\{\frac{1}{C_{t+1}}\frac{1}{(1+\rho)}\left[a\,U_{t+1}A_{t+1}\,K_{t+1}^{a-1}\,L_{t+1}^{b} + \frac{A_{t+1}}{B_{t+1}}(1-\delta)\right]\right\}.$$

(A12) is the intratemporal first order condition from choice of L and (A13) is the intertemporal first order condition from choice of K.

Step 4: The Dynamic Equations of Motion for the Economy

The next step is to combine the equilibrium values of the externality parameters, given by equations (A10) and (A11), with the decision rules (A12) and (A13), the definition of the private ppf (A3), the definition of the allocation variable (A9), and the capital accumulation equation (A4) to arrive at a set of five equations in the five unknowns, K,L,C,I and S.

Equation (A15) The Labor Market Equation

First, replacing A in (A12) from (A10):

(A14) $$L_t^{\chi}\,C_t = b\,U_t\,\frac{K_t^{\alpha}\,L_t^{\beta-1}}{S_t^{v-1}}.$$

Using the definition of S (equation (A9)) we can write this equation more compactly:

(A15) $$L_t^{1+\chi} = bS_t.$$

Equation (A18) The Social PPF

Using (A10), and (A11) to replace A and B in (A3) leads to the equation:

(A16) $$C_t + \frac{I_t}{(S_t-1)^{v-1}} = \frac{U_t\,K_t^{\alpha}\,L_t^{\beta}}{S_t^{v-1}}.$$

We may also take the ratio of (A1) to (A2) and, using equations (A8) and (A9) to replace A/B and μ by functions of S we can find a relationship between C/I and S:

(A17) $$\frac{C_t}{I_t} = \frac{1}{(S_t-1)^{v}}.$$

Using (A17) to eliminate S from (A16) leads to an equation that we refer to as the social ppf:

(A18) $$C_t^{1/v} + I_t^{1/v} = K_t^{\alpha/v}\,L_t^{\beta/v}\,U_t^{1/v}.$$

Notice that for the case of v=1 this equation collapses to the standard linear ppf of the one sector model.

Equation (A19) The Definition of the Sectoral Allocation Parameter

Equation (A9) – which we restate – defines the variable S:

(A19) $$S_t = \frac{K_t^{\alpha/v} L_t^{\beta/v} U_t^{1/v}}{C_t^{1/v}}.$$

Equation (A20) The Capital Accumulation Identity

Similarly (A4) – restated below – defines capital accumulation:

(A20) $$K_{t+1} = K_t(1-\delta) + I_t.$$

Equation (A22) The Stochastic Euler Equation

Replacing A and B in (A13), using equations (A10) and (A11), leads to:

(A21) $$\frac{1}{C_t}\frac{1}{(S_t-1)^{v-1}} = E_t\left\{\frac{1}{C_{t+1}}\frac{1}{(1+\rho)}\left[a\,U_{t+1}\frac{K_{t+1}^{\alpha-1}L_{t+1}^{\beta}}{S_{t+1}^{v-1}} + \frac{(1-\delta)}{(S_{t+1}-1)^{v-1}}\right]\right\}.$$

Using equation (A9) (the definition of S) this equation can be written in the form:

(A22) $$\frac{1}{C_t}\frac{1}{(S_t-1)^{v-1}} = E_t\left\{\frac{1}{C_{t+1}}\frac{1}{(1+\rho)}\frac{1}{(S_{t+1}-1)^{v-1}}\left[a\,S_{t+1}(S_{t+1}-1)^{v-1}\frac{C_{t+1}}{K_{t+1}}+(1-\delta)\right]\right\}$$

Equations A15, A18, A19, A20 and A22 represent five dynamic equations in five variables that define the behavior of the model.

Step 5: Computing the Stationary State

Since the system is non-linear our approach is to linearize around the stationary state of the non-stochastic model. To solve for this stationary state we set U_t equal to U and solve for the steady state. In the following analysis we let variables without time subscripts represent stationary values. From (A20) we have:

(A23) $$I = \delta K,$$

and from (A22), (A23) and (A17):

(A24) $$1 = \frac{1}{(1+\rho)}\left[a\frac{S}{(S-1)}\delta + 1-\delta\right],$$

which can be rearranged to give:

(A25) $$S = \frac{\rho+\delta}{\rho+\delta-a\delta}.$$

Given the steady value of S, L is found from (A15):

(A26) $L = (bS)^{1/(1+\chi)}$

From (A19) we have that

(A27) $$S = \frac{K^{\alpha/v} L^{\beta/v} U^{1/v}}{C^{1/v}},$$

and from (A23) in conjunction with (A17):

(A28) $$\frac{\delta K}{C} = (S-1)^{v}.$$

Using (A28) to replace K in (A27) leads to:

(A29) $$S = \left[\frac{(S-1)^{v} C}{\delta}\right]^{\alpha/v} \frac{(bS)^{\beta/v(1+\chi)} U^{1/v}}{C^{1/v}}$$

which can be arranged to give C as a function of S:

(A30) $$C = \frac{(S-1)^{\alpha v/(1-\alpha)} S^{(\beta-v(1+\chi))/(1+\chi)(1-\alpha)} b^{\beta/(1+\chi)(1-\alpha)} U^{1/(1-\alpha)}}{\delta^{\alpha/(1-\alpha)}}.$$

The steady state value of K follows from (A23) and (A17) which together imply:

(A31) $$K = \frac{C(S-1)^{v}}{\delta}.$$

Finally, from (A23) it follows that:

(A32) $I = \delta K.$

Equations A25, A26, A30, A31 and A32 can be used recurively to compute the values of the steady values of S,L,C,K and I for any given specification of parameters.

Step 6: Taking Logarithms of the State Equations

In the following analysis we compute a log linear specification of equations A15, A18, A19, A20, and A22 by computing a first order Taylor series expansion of these equations around the steady state of the non-stochastic model. We use the symbols k_t, c_t, i_t, s_t and z_t to refer to the natural logarithms of K_t, C_t, I_t, S_t and L_t. Taking logs of A15, A18, A19, A20 leads to the equations:

(A33) $$(1+\chi)z_t = \log(b) + s_t,$$

(A34) $$\log\left(C_t^{1/v} + I_t^{1/v}\right) = \frac{\alpha}{v}k_t + \frac{\beta}{v}z_t + \frac{1}{v}u_t,$$

(A35) $$s_t = \frac{\alpha}{v}k_t + \frac{\beta}{v}z_t + \frac{1}{v}u_t - \frac{1}{v}c_t,$$

(A36) $$k_{t+1} = \log\left((1-\delta)K_t + I_t\right).$$

For the time being we restate equation (A22) leaving the equation in levels.

$$\text{(A37)}\quad \frac{1}{C_t(S_t-1)^{v-1}} = E_t\left\{\frac{1}{C_{t+1}(S_{t+1}-1)^{v-1}}\frac{1}{(1+\rho)}\left[a S_{t+1}(S_{t+1}-1)^{v-1}\frac{C_{t+1}}{K_{t+1}}+(1-\delta)\right]\right\}$$

Step 7: Writing the Model in LogLinear Form

In the following derivations of the loglinear model we use the notation k,s,i,c,z,u – where lower case letters without time subscripts refer to logs of steady state values. We also use the notation dk,ds,di,dc,dz and du to refer to logarithmic deviations from the steady state and we make use of the following approximation:

$$\text{(A38)}\qquad dx_t = \log(x_t) - \log(x) \cong \left(\frac{X_t - X}{X}\right).$$

Equations (A33) and (A35) are already loglinear. Writing these equations as log deviations from steady states leads to equations (A39) and (A40):

$$\text{(A39)}\qquad (1+\chi)dz_t = ds_t,$$

$$\text{(A40)}\qquad ds_t = \frac{\alpha}{v}dk_t + \frac{\beta}{v}dz_t + \frac{1}{v}du_t - \frac{1}{v}dc_t.$$

Next we take a first order Taylor series approximation of equations (A34), (A36), and (A37) around the point k,c,i,s,z,u. We collect the following results that we will use to define coefficients in the log approximation of (A34).

$$\text{(A41)}\qquad \begin{aligned}\log\left(C_t^{1/v} + I_t^{1/v}\right) &= \log\left(C^{1/v} + I^{1/v}\right) + \left(\frac{C^{1/v}}{C^{1/v}+I^{1/v}}\right)\frac{1}{v}\frac{(C_t - C)}{C}\\ &+ \left(\frac{I^{1/v}}{C^{1/v}+I^{1/v}}\right)\frac{1}{v}\frac{(I_t - I)}{I} + O(x^2)\end{aligned}$$

But from (A18) and (A9):

$$\text{(A42)}\qquad \frac{C^{1/v}}{C^{1/v}+I^{1/v}} = \frac{C^{1/v}}{K^{\alpha/v}L^{\beta/v}U^{1/v}} = \frac{1}{S}.$$

It follows that

$$\text{(A43)}\qquad \frac{I^{1/v}}{C^{1/v}+I^{1/v}} = \frac{S-1}{S}.$$

Taking a Taylor series expansion of (A34) (dropping second order terms) leads to the equation:

$$\text{(A44)}\qquad \frac{1}{vS}dc_t + \frac{(S-1)}{vS}di_t = \frac{\alpha}{v}dk_t + \frac{\beta}{v}dz_t + \frac{1}{v}du_t.$$

Taking a first order expansion of the right side of equation (A36) leads to the expression:

(A45) $$\log\left((1-\delta)K + I\right) + \left(\frac{(1-\delta)K}{(1-\delta)K+I}\right)\frac{(K_t - K)}{K} + \left(\frac{I}{(1-\delta)K+I}\right)\frac{(I_t - I)}{I} + O(x^2)$$

Using the fact that I=δK we can use (A45) to write the loglinear approximation to (A36):

(A46) $$dk_{t+1} \equiv (1-\delta)dk_t + \delta di_t.$$

To find the loglinear approximation to (A37) we note that if X represents the vector of state variables then (A37) takes the form:

(A47) $$f(X_t) = E_t \{g(X_{t+1})\}.$$

We approximate both sides of (A47) with a Taylor series expansion:

(A48) $$\text{const} + f_x \frac{(X_t - X)}{X} = E_t \left\{ g_x \frac{(X_{t+1} - X)}{X} + O(x^2) \right\}$$

and make use of the approximation (A38). Assuming that all noise is small and bounded, and that the dynamical system remains in the neighborhood of the steady state, this approximation will be good. The better the assumptions, the better the approximation. Using this approximation we can write (A37) in the form:

(A49) $$-dc_t - (v-1)dp_t = E_t \{-dc_{t+1} - (v-1)dp_{t+1} + dq_{t+1}\}$$

where dp and dq are log differences of the variables P and Q, defined below:

(A50) $$P_t = (S_t - 1),$$

(A51) $$Q_{t+1} = [R_{t+1} + (1-\delta)],$$

and R is given by

(A52) $$R_{t+1} = a S_{t+1} (S_{t+1} - 1)^{v-1} \frac{C_{t+1}}{K_{t+1}}.$$

Using (A38) we can compute the follwing log difference approximations to (A50), (A51) and (A52)

(A53) $$dp_t \cong \frac{S}{(S-1)} ds_t,$$

(A54) $$dq_{t+1} \cong \frac{R}{(R+1-\delta)} dr_{t+1},$$

(A55) $$dr_{t+1} \cong \frac{(vS-1)}{(S-1)} ds_{t+1} + dc_{t+1} - dk_{t+1}.$$

Using these approximations, (A49) becomes:

$$-dc_t - \frac{S}{(S-1)}(v-1)ds_t =$$

(A56) $$E_t\left\{-dc_{t+1} - \frac{S}{(S-1)}(v-1)ds_{t+1} + \frac{(vS-1)}{S-1}\phi\, ds_{t+1} + \phi\, dc_{t+1} - \phi\, dk_{t+1}\right\},$$

where ϕ is defined as:

(A57) $$\phi = \frac{R}{R+(1-\delta)}.$$

On the assumption that the steady state equilibrium of the model is indeterminate – we search for solutions generated by the system of equations: (A39), (A40) (A44) (A46) plus:

$$-dc_t - \frac{S}{(S-1)}(v-1)ds_t =$$

(A57) $$-dc_{t+1} - \frac{S}{(S-1)}(v-1)ds_{t+1} + \frac{(vS-1)}{S-1}\phi\, ds_{t+1} + \phi\, dc_{t+1} - \phi\, dk_{t+1} + dw_{t+1},$$

where dw_{t+1} is defined by the equation:

$$dw_{t+1} = E_t\left\{-dc_{t+1} - \frac{S}{(S-1)}(v-1)ds_{t+1} + \frac{(vS-1)}{S-1}\phi\, ds_{t+1} + \phi\, dc_{t+1} - \phi\, dk_{t+1}\right\} -$$

(A58) $$\left\{-dc_{t+1} - \frac{S}{(S-1)}(v-1)ds_{t+1} + \frac{(vS-1)}{S-1}\phi\, ds_{t+1} + \phi\, dc_{t+1} - \phi\, dk_{t+1}\right\}$$

We search for equilibria of the model by specifying an arbitrary time series process for w_{t+1} – which we refer to as a belief shock.

Step 8: Restating the Equations of the Model

We collect together the equations of the model, together with an additional equation that allows for autocorrelation in the productivity shock U. We refer to the innovation in the log productivity shock as de_t and we let λ represent the autocorrelation parameter.

(A59) $(1+\chi)dz_t - ds_t = 0,$

(A60) $ds_t - \frac{\alpha}{v}dk_t - \frac{\beta}{v}dz_t - \frac{1}{v}du_t + \frac{1}{v}dc_t = 0,$

(A61) $\frac{1}{vS}dc_t + \frac{(S-1)}{vS}di_t - \frac{\alpha}{v}dk_t - \frac{\beta}{v}dz_t - \frac{1}{v}du_t = 0,$

(A62) $dk_{t+1} - (1-\delta)dk_t - \delta di_t = 0,$

$$-dc_t - \frac{S}{(S-1)}(v-1)ds_t$$

(A63) $$+dc_{t+1} + \frac{S}{(S-1)}(v-1)ds_{t+1} - \frac{(vS-1)}{S-1}\phi ds_{t+1} - \phi dc_{t+1} + \phi dk_{t+1} - dw_{t+1} = 0$$

(A64) $du_{t+1} = \lambda du_t + de_{t+1}$

Step 9: Computing Equilibria

In this step we show how to simulate the model. For compactness we write the model in matrix form by defining three new vectors:

(A65) $$y_t = \begin{bmatrix} dc_t \\ dk_t \\ du_t \end{bmatrix}, \qquad x_t = \begin{bmatrix} ds_t \\ di_t \\ dz_t \end{bmatrix}, \qquad \varepsilon_t = \begin{bmatrix} de_t \\ dw_t \end{bmatrix}$$

Using this notation we can write the three static equations, (A59) (A60) and (A61) in the form:

(A66) $A_1 y_t + A_2 x_t = 0,$

where the matrices A_1 and A_2 are defined as follows:

(A67) $$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1/v & -(\alpha/v) & -(1/v) \\ 1/(vS) & -(\alpha/v) & -(1/v) \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 & (1+\chi) \\ 1 & 0 & -(\beta/v) \\ 0 & (S-1)/vS & -(\beta/v) \end{bmatrix}.$$

Similarly, we may write the dynamic equations, (A62) and (A63) in the form:

(A68) $B_1 y_{t+1} + B_2 x_{t+1} + B_3 y_t + B_4 x_t + B_5 \varepsilon_{t+1} = 0$

where the matrices B_1, B_2, B_3, B_4, and B_5, are defined below:

$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ (1-\phi) & \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ S(v-1)/(S-1) - (vS-1)\phi/(S-1) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\text{(A69)}\quad B_3 = \begin{bmatrix} 0 & -(1-\delta) & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -\lambda \end{bmatrix}, \qquad B_4 = \begin{bmatrix} 0 & -\delta & 0 \\ -S(v-1)/(S-1) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B_5 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Assuming that A_2 is of full rank we can write this system as:

(A70) $$x_t = J_1 y_t,$$

(A71) $$y_{t+1} = Q_1 y_t + Q_2 \varepsilon_{t+1},$$

where J_1, Q_1, and Q_2 are defined as:

(A72) $$J_1 = -(A_2)^{-1} A_1, \qquad Q_1 = -(B_1 + B_2 J_1)^{-1}(B_3 + B_4 J_1),$$
$$Q_2 = -(B_1 + B_2 J_1)^{-1} B_5.$$

Step 10: Calibration

The standard deviations and correlations of $\{x_t\}$ and $\{y_t\}$ can easily be computed analytically given the matrices Q_1, Q_2 and the variance-covariance matrix of the innovations $\{\varepsilon_t\}$. For purposes of calibration we set the capial share, a, to 0.35 and the labor share, b, to 0.65, the quarterly depreciation rate, δ, to 0.025, the quarterly discount rate, ρ, to 0.01, the inverse elasticity of labor supply, χ, to 0 implying linear preferences in leisure, and the persistence parameter in the technology shock, λ, equal to 0.95. As mentioned above, we will assume that the innovation to the technology shock and the sunspot are correlated. This is a simple way to obtain procyclical consumption.[1] For simpicity we assume that the innovation to technology,

[1] We can also get procyclical consumption that is positively correlated with contemporaneous output without correlated shocks. This is possible because technology shocks lead to changes in capital and wealth which then tend to pull consumption along.

$\{e_t\}$, and the sunspot, $\{w_t\}$,are driven by the same stochastic process and are in fact identical and perfectly correlated. The standard deviation of the common shock is 0.09 and is calibrated to match the standard deviation of output, which is taken as 1.76. Below we report the results of our calibration.[2]

	Consumption	Investment (pI)	Hours	Productivity (Wages)
rstd	0.7420480	3.458355	0.8949918	0.7420480
corr	0.5051054	0.831482	0.6985400	0.5051054

rstd = standard deviation of variable/standard deviation of output

corr = correlation with output

The steady state ratios are as follows:Consumption/Output =0.80, Investment/Output =0.20.

While the moments reported above certainly do not represent an exact match to the data, they are not implausibly different from the data either. Introducing an HP filter may also further improve the match. Below we present some pictures from simulated time-series for consumption, output and investment.

[2] We should note that these analytically computed moments have not been adjusted for a Hodrick-Prescott filter. It should be easy to modify the computations to incorprate the effects of the filter into the computations.

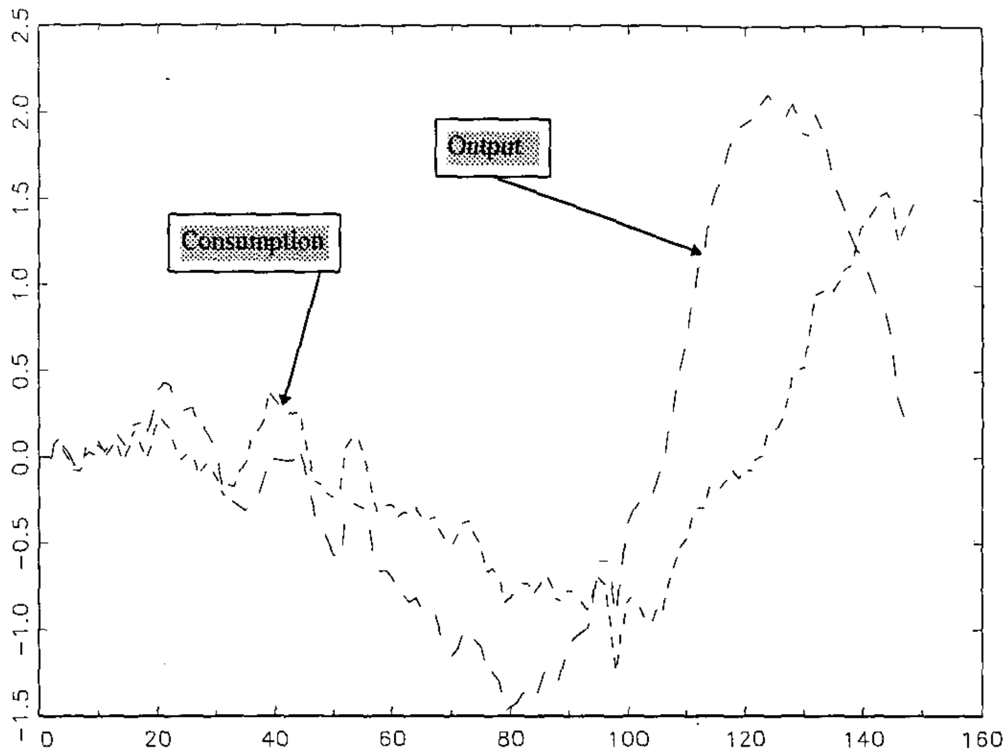

FIGURE A1

Percentage deviations from the steady state: consumption and output

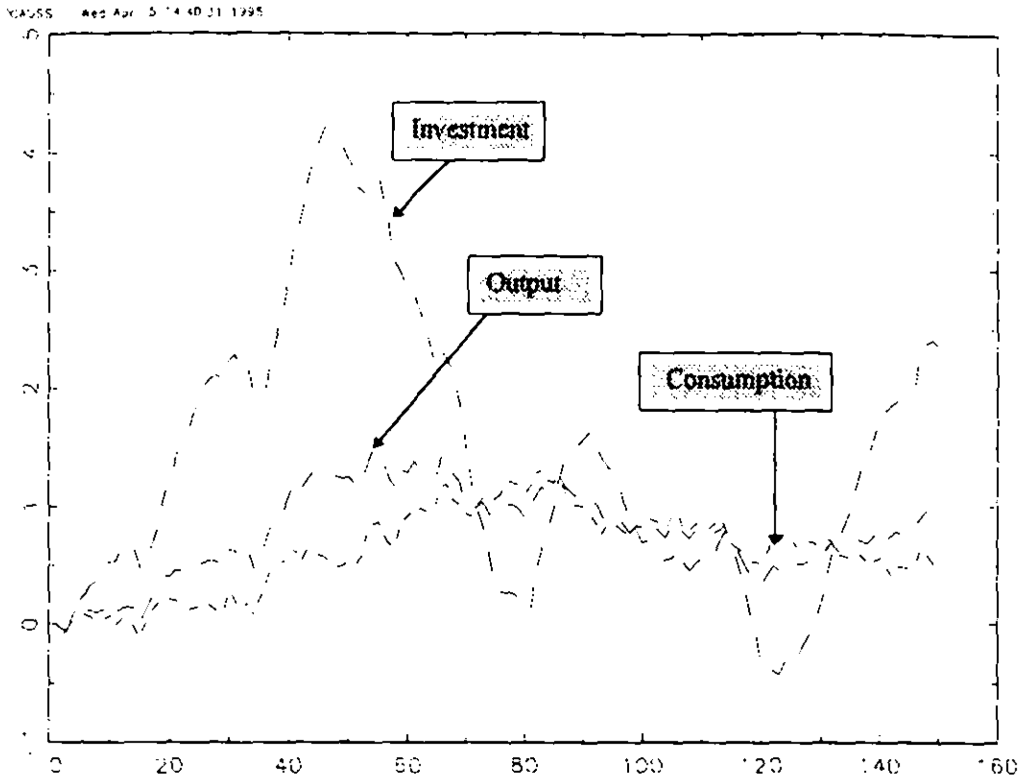

FIGURE A2

Percentage deviations from steady state: Consumption, Output and Investment

We should also note that for our calibration the Q_1 matrix in equation (A71) above has complex roots of less than that unit modulus. They generate impulse response functions of investment, consumption and output to a technology or sunspot shock that are hump-shaped, as has also been pointed out in Farmer and Guo (1994).

Finally, we note that we can dispense with the technology shock altogether in our calibration and still have a reasonable match with some of the moments in the data. Below we report the results of our calibration where the only shock to the economy is a sunspot shock with a standard deviation of 09 as before.

	Consumption	Investment (pI)	Hours	Productivity (Wages)
rstd	0.7540204	3.908807	1,040209	0.7540204
corr	0.3226123	0.8366316	0.7274922	0.3226123

rstd = standard deviation of variable/standard deviation of output

corr = correlation with output

We see that without technology shocks consumption is less correlated with output and investment is more variable than before. As pointed out in footnote 1 of the appendix, some positive correlation nevertheless remains due to the movements in capital..