FEDERAL RESERVE BANK OF MINNEAPOLIS

Research Division

WORKING PAPER
No. 794

Permanent Primary Deficits, Idiosyncratic Long-Run Risk, and Growth

November 2022

Amol Amol
University of Minnesota

Erzo G.J. Luttmer
University of Minnesota and Federal Reserve Bank of Minneapolis

DOI: https://doi.org/10.21034/wp.794
Keywords: Long-run idiosyncratic risk; Incomplete markets; Endogenous growth; Public debt
JEL classification: E6, H6, O4

Abstract

We consider an economy with perpetual youth and inelastic labor supply that grows endogenously. Consumers are subject to idiosyncratic capital accumulation risk and markets are incomplete. The government purchases consumption goods, makes transfers in the form of baby bonds, and it can use consumption and wealth taxes. The wealth distribution is given in closed form. When the intertemporal elasticity of substitution ε is equal to 1, the government can run a permanent primary deficit, up to a finite upper bound, if the coefficient of relative risk aversion is high enough and the factor share of labor is not too close to 1. This causes the risk-free rate r to be below the growth rate g of the economy. But the government can implement Pareto improvements when $r - g$ does not exceed zero by enough. If $\varepsilon \neq 1$, then there may not be an upper bound on the permanent primary deficits of the government. If $\varepsilon \in (0, 1)$, this happens when the economy is relatively unproductive, and then taking deficits to be very large makes all consumers worse off. If $\varepsilon \in (1, \infty)$, very large deficits are possible if the economy is sufficiently productive, and then they imply unbounded Pareto improvements.[1]

[1]We thank seminar participants at USC, UCLA, and the Federal Reserve Bank of Minneapolis for useful comments on earlier versions of this paper. An appendix with additional proofs is available upon request.

Contents

1 Introduction

What are the effects on growth, inequality, and welfare, of persistent and possibly large primary government deficits?

In this paper we present a stylized model of an endogenously growing economy in which it is possible to give detailed analytical answers to this question. In our model, consumers die randomly, as in Blanchard [1985] and Yaari [1965], and there is a constant flow of newborn consumers. Consumers have access to their own individual capital accumulation technologies that are subject to idiosyncratic Brownian shocks. Life span risk can be insured but otherwise markets are incomplete. Capital can be traded, but there are no markets to hedge the risky returns that come with accumulating capital. Nobody can earn the returns to holding capital without also being exposed to idiosyncratic risk.

The individual capital accumulation technologies are linear in capital. Consumption is produced using a Cobb-Douglas technology that uses capital and labor. The economy can be viewed as a special case of the two-sector economy of Uzawa [1965], but with a capital producing sector that is extremely capital intensive (linear in capital only), and a consumption sector that is relatively labor intensive. This implies a production possibility frontier for consumption and new capital that is strictly concave, resulting in an endogenously determined price of capital that declines more rapidly the faster the economy grows.[1 2]

Consumers have Epstein and Zin [1989] preferences. This allows for enough risk aversion, independently of the intertemporal elasticity of substitution $\varepsilon \in (0, \infty)$. This elasticity, and especially the sign of $1 - \varepsilon$, plays a central role in evaluating the effects of government policy in this economy.[3]

The government issues nominal securities and prices are flexible. The government can purchase consumption goods, make transfers to newborn consumers (baby bonds), and impose linear taxes on consumption and wealth. Given stable government policies, aggregates in this economy are always on a balanced growth path. The distribution of

[1]Rebelo [1991] already mentions a deterministic version of this model. Jones and Manuelli [1992] use it to avoid stagnation in an overlapping generations economy. See Galor [1992] for a detailed analysis of an Uzawa economy with overlapping generations. Luttmer [2012] shows that the competitive quality ladder model of Boldrin and Levine [2010] converges to this economy as the ladder steps become small. As Greenwood and Jovanovic [2001] emphasize, a falling price of capital goods relative to consumption goods is a central feature of the data.

[2]As in Jones and Manuelli [2005], allocations would be efficient if markets were complete and consumers were infinitely lived. See Atkeson and Burstein [2019] for a recent examination of tax policy in economies in which externalities are important for growth.

[3]For divergent views on whether the empirical evidence points to $\varepsilon \in (0, 1)$ or to $\varepsilon \in (1, \infty)$, see, respectively, Yogo [2004] and Schorfheide, Song, and Yaron [2018].

wealth at a given point in time can be anything, but it will eventually converge to a stationary distribution. This stationary distribution is a double Pareto distribution. Taxes and transfers, and the factor share of labor are important determinants of how thick the tail of the wealth distribution will be.[4]

The case of a unit intertemporal elasticity of substitution is quite special, but also extremely tractable. With a government that runs a balanced budget and makes no transfers, our incomplete markets economy can have pure bubble assets (no dividends, strictly positive prices) if and only if there is enough idiosyncratic risk and the labor share is less than the ratio of the death rate over the sum of the death rate and the subjective discount rate. Even if there is a lot of idiosyncratic risk, this rules out pure bubble assets if consumers are very impatient or very long-lived. Bubble assets can emerge to improve risk sharing, but only if the value of the safe claims to labor is not too high already.[5] In a pure bubble equilibrium, $r-g$ is equal to zero. When a pure bubble equilibrium is possible, the economy will also have a no-bubble equilibrium with $r-g<0$. Conversely, $r-g<0$ in a no-bubble equilibrium implies the existence of a pure bubble equilibrium. Bubble assets can include not only government securities but also other useless pieces of paper. As in Kareken and Wallace [1981], the relative price of competing bubbles is indeterminate.[6]

Starting from a balanced budget policy that allows for a bubble equilibrium, the government can, within limits, permanently lower consumption taxes or increase transfers to newborn consumers. If the intended deficits are not too large, we show that there is a steady state equilibrium in which the government can simply cover these deficits by selling more nominal securities, forever. A steady state requires a stable real value of government securities relative to the size of the economy, and this forces $r-g<0$. An unforeseen change in government policy can cause a one-time jump in the price level. Other than that, primary deficits have nothing to do with inflation in our economy.

How large permanent primary deficits can be depends very much on how these deficits are used. For reasonable parameters, the upper bounds on primary deficits will be on the

[4]Toda [2014] uses a standard AK economy, without a fixed factor, and with incomplete markets and Epstein-Zin preferences to generate a wealth distribution that is double Pareto. Benhabib and Bisin [2010] developed a closely related model of the distribution of wealth that relies on overlapping generations and perpetual youth.

[5]Aoki, Nakajima, and Nikolov [2014] show the possibility of a bubble asset in an AK economy with idiosyncratic risk but no fixed factor. As in their paper, occasionally binding borrowing constraints play no role in our economy, unlike in Harrison and Kreps [1978] or Santos and Woodford [1997]. See Miao [2014] for a more recent discussion of models of asset price bubbles.

[6]If there is a fixed supply of private sector long-lived assets that deliver no dividends, then the value of these assets will shrink at the rate $r-g<0$ relative to aggregate consumption. And they do not restrict government policy. But this presumes there is no private sector entity that issues more of these assets, just like the government does. Throughout this paper, we will rule this out by assumption and abstract from private-sector long-lived assets altogether.

order of a few percent of household consumption expenditures if the government only cuts taxes or only increases transfers. But if the government uses the right combination of primary deficits and consumption taxes to fund baby bonds, then the primary deficit of the government has an upper bound that approaches 100% of household consumption expenditures when idiosyncratic risk or risk aversion becomes large. This requires large consumption taxes and even larger transfers of baby bonds. Our assumption that transfers are in the form of baby bonds also matters a great deal. A government that instead makes transfers in the form of a universal basic income again faces a tight constraint on the size of its permanent primary deficits. Such a policy reduces the demand for safe government securities, which in turn lowers the upper bound on how large deficits can be. The government is a large agent whose policies affect the economy not just via the size of its deficits. There is no notion of "fiscal space" that is independent of how the government uses that space.

We show that increases in baby bonds financed by higher consumption taxes are always good for long-run growth when the government runs a primary surplus. This is already true in a complete markets economy with perpetual youth. Holding taxes fixed and simply increasing transfers of baby bonds also increases growth, even when the government begins to run a deficit, provided that the equilibrium changes continuously when the primary surplus of the government switches to a deficit. These growth effects are clearly beneficial for generations that will be born far enough into the future. But the associated declines in current aggregate consumption and increases in risk exposure are sufficiently painful that all consumers already alive are hurt by increased transfers of baby bonds. On the other hand, any policy that reduces growth can never be a Pareto improvement because it hurts generations that will be born sufficiently far into the future. If the government can only change its policy on transfers, then there is no way to change policy and make everyone better off.

But if the government can vary both its transfers of baby bonds and consumption taxes, then the government can separately target both the aggregate growth rate of the economy and the amount of idiosyncratic risk consumers are exposed to. In that case, we prove that an equilibrium with $r - g < 0$ implies the existence of alternative policies that lead to allocations that Pareto dominate the equilibrium allocation with $r - g < 0$. So permanent primary deficits can never be efficient when the government has access to both baby bonds and consumption taxes. More strongly, the fact that markets are incomplete actually implies that there will still exist Pareto improving policies if $r - g > 0$ and close enough to zero.[7]

[7]Others have emphasized that, in certain settings, government policies that lead to $r - g < 0$ can be

Using a combination of very large transfers and even larger consumption taxes, the government can actually implement any feasible aggregate growth rate and at the same time eliminate almost all idiosyncratic risk and achieve allocations that are approximately Pareto efficient. Such policies are favored by consumers who are already alive. But even though the current generation of newborn consumers will be the first recipients of these large transfers, they strictly prefer policies that do not eliminate almost all idiosyncratic risk. When the equilibrium exposure to idiosyncratic risk is already small, the direct benefits of further reductions in this exposure are of second order. At the same time, reductions in idiosyncratic risk exposure do increase the risk-free rate, linearly. This increases individual consumption growth for everyone, without changing aggregate consumption or aggregate consumption growth. As a matter of accounting, this requires a reduction in newborn consumption relative to aggregate consumption, which hurts newborn consumers.

This intergenerational conflict over the desirability of eliminating all idiosyncratic risk disappears when the government can also use a wealth tax. Together with transfers of baby bonds, this instrument allows the government to reduce individual consumption growth rates without distorting the aggregate growth rate of the economy. This allows the government to increase newborn wealth relative to aggregate wealth, independently of the aggregate growth rate and the equilibrium exposure to idiosyncratic risk. All generations then prefer policies that approximately eliminate all exposure to idiosyncratic risk. Wealth taxes are still inconsistent with Pareto efficiency. But because they raise newborn wealth relative to aggregate wealth, current and future newborn generations prefer strictly positive wealth taxes.

The $\varepsilon = 1$ economy is very well behaved. A key reason is that the homogeneous of degree 1 version of utility is always finite and positive, no matter how productive or unproductive the individual capital accumulation technologies may be. This is very much not the case when ε differs from 1.

If $\varepsilon \in (0, 1)$ instead, and the capital accumulation technologies are not productive enough, then a central planner can only deliver zero utility to everyone but a zero measure of consumers. Above a certain productivity threshold, the economy will have a complete markets equilibrium. But with no government securities outstanding, the fact that individual consumption trajectories will have to be risky means that the economy still does not have an incomplete markets equilibrium. Once productivity rises above an-

improved upon. Abel and Panageas [2022] and Hellwig [2021] do so in the context of two-period OLG economies with aggregate risk, and Brumm, Feng, Kotlikoff, and Kubler [2022] and Kocherlakota [2022] do so in the context of idiosyncratic risk and incomplete markets. In the $\varepsilon = 1$ setting we describe, $r - g = 0$ is still a sign of Pareto inefficiency.

other threshold, the economy does also have an incomplete markets equilibrium. In between these two productivity thresholds, the economy has a pure bubble equilibrium, in which government securities trade at a positive price even though its primary surpluses are always zero. Perhaps more strikingly, precisely in between these two productivity thresholds, holding fixed taxes, there is no bound on how large government transfers can be. But we show that very large transfers are the worst possible policy a government can choose. It drives everyone's utility back down to zero.

On the other hand, if $\varepsilon \in (1, \infty)$, then a central planner can increase everyone's utility without bound if the productivity of the capital accumulation technologies is above a certain threshold. In this region, the government can ensure the existence of a complete markets equilibrium by choosing a large enough tax on wealth. By choosing this wealth tax just large enough, the government can increase utilities without bound. But even without wealth taxes, an economy that is too productive to have a complete markets equilibrium will still have an incomplete markets equilibrium if its productivity does not rise above another threshold. The idiosyncratic risk to which consumers are exposed is enough to keep utility finite. In between these two threshold, we show that the government can increase transfers without bound. And we show that these large transfers provide another way in which the government can increase utilities without bound, one that does not rely on any form of taxation. We show, by example, that when unbounded utilities are possible, increasing transfers to newborn consumers starting from a balanced budget baseline can actually be an immediate Pareto improvement, not just when transfers become large.

To dissect this unbounded utility result, abstract, for simplicity, from government purchases and any form of taxation, so that household consumption expenditures and consumption are the same. When transfers become large relative to consumption, government deficits become large relative consumption, and it must be that for any given $r - g < 0$ the steady state market value of government securities also becomes large relative to consumption. Since the portfolio share of government securities is bounded above by 1, this means that aggregate wealth must also become large relative to aggregate consumption. That is, the consumption-wealth ratio must go to zero. The consumption-wealth ratio is a budget share, and for CES preferences with an elasticity $\varepsilon \in (1, \infty)$, a budget share that converges to zero means utility going to infinity relative to a quantity consumed. As transfers become large, we show that the growth rate of the economy converges to its technological upper bound, as most capital is used to produce new capital rather than consumption. The marginal product of capital in the consumption sector, and therefore the price of capital, becomes very high. This stabilizes the market value of the aggregate capital stock relative to the value of government securities. Risk-adjusted individual consumption growth converges to the upper bound of the range where utility is finite. These two limiting conditions —aggregate growth at its technological upper bound, and individual risk-adjusted consumption growth at the boundary where utility explodes— determine a strictly negative limiting value of $r-g$ and a limiting portfolio share of capital that is in $(0,1)$. In sharp contrast to our result for the $\varepsilon=1$ economy, negative $r-g$ in this economy is entirely consistent with government policies that approximate the best they can be.

The unbounded utility result is intimately related to our perpetual youth assumption. Taken literally, the only possible interpretation is that households are dynasties of altruistically linked consumers, as in Weil [1989]. Implicitly, we assume dynasties die and new dynasties are born when altruistic links break down. The large-transfer policy that leads to unbounded utilities then means that new dynastic households must receive large transfers.[8] We also consider an economy in which households cannot last beyond a definite age T. We prove that it is possible to construct fiscal policies for the finite-T economy that approximate the utilities in the perpetual youth economy as T becomes large.

Related Literature Going back to Samuelson [1958] and Diamond [1965], there is, of course, a vast literature on economies with real interest rates that are below their growth rates. Blanchard [2019] and the facts of the US economy, and of other advanced economies as well, have triggered a renewed interest in the topic.[9] The following discussion focuses on the most closely related work that emphasizes idiosyncratic risk and market incompleteness.

Brunnermeier, Merkel and Sannikov [2022] and Reis [2021] both describe AK economies with infinitely-lived consumers, Brownian uncertainty, and incomplete markets, in which the rates of return on safe government securities is low.[10] But there is no fixed factor of production in their models. In our economy, labor is a fixed factor and consumers can borrow against their labor income. This creates a private-sector safe asset that makes $r-g<0$ impossible when consumers are infinitely lived. Some type of overlapping generations structure is essential to make unbacked government securities trade at a positive price. Our baseline choice is a model of perpetual youth, but we show that transfers not

[8] Unbounded utilities are also possible when the government sets transfers to zero for everyone and generates large deficits by paying large consumption subsidies.

[9] For example, Barro [2021] and Mehrotra and Sergeyev [2021] obtain $r-g<0$ in representative agent economies with aggregate risk. In Abel and Panageas [2022], Blanchard and Weil [2001], Brumm, Feng, Kotlikoff, and Kubler [2021] and Hellwig [2021] there is aggregate risk that cannot be shared between overlapping generations of two-period lived consumers.

[10] In addition, Brunnermeier, Merkel and Sannikov [2022] have aggregate risk, and Reis [2021] has entrepreneurs who cannot borrow more than certain fraction of their returns to capital.

paid for by taxes can also be Pareto improving when there are overlapping generations of consumers with bounded life spans. In Brunnermeier, Merkel and Sannikov [2022], the price of capital is endogenous because new capital is produced from existing capital and final output, using a constant returns technology that exhibits diminishing returns to these two inputs. This is a classic form of adjustment cost that creates protracted transitions when policy changes. As a result, making explicit welfare statements is beyond what is analytically tractable. In our economy, the price of capital is endogenous because its marginal product in the consumption sector is decreasing in the amount of capital used to produce consumption. Similar to an AK economy with a linear consumption sector, this means that the aggregate economy (though not the underlying wealth distributions) will switch to a new balanced growth path instantaneously. This prevents us from analyzing aggregate transitions, but it does allow us to describe the effects of permanent changes in fiscal parameters explicitly. The welfare consequences of a policy change can be summarized by only three statistics: the utility of the representative consumer already alive at the time of the policy change, the utility of the new generation born at the time of the policy change, and the effect of the policy change on long-run growth. In particular, fiscal policies that hurt growth can never be Pareto improving in our economy. Another important difference with Brunnermeier, Merkel and Sannikov [2022] and Reis [2021] is that we do not restrict attention to consumer preferences with a unit intertemporal elasticity of substitution. This is essential for making welfare improving large deficits a robust possibility.

Incomplete markets also play an important role in the Aiyagari-Bewley-Huggett economy with a neoclassical technology described by Aguiar, Amador and Arellano [2021]. They characterize the types of fiscal policies for the government that can result in Pareto improvements. In their economy, equilibrium allocations can be dynamically inefficient. This is ruled out by our AK-style Uzawa technology. Introducing idiosyncratic labor income risk in our environment reduces the private-sector supply of safe assets and is likely to increase the size of the permanent primary deficits that are consistent with equilibrium.

Kocherlakota [2022] considers an Aiyagari-Bewley-Huggett economy in which consumers face a near-zero probability of a highly adverse outcome. Due to a precautionary savings motive strengthened by this tail risk, there is a strong demand for risk-free government bonds. There is an upper bound on the value of government debt in this economy, and that upper bound is attained in a steady state with $r - g = 0$. As the probability of the state with the highly adverse outcome goes to zero and the marginal utility of consumption in this disaster state goes to infinity, this upper bound also goes to infinity. In our economy, the uninsurable idiosyncratic investment risk faced by households grows without bound over long horizons. If dynastic households can survive with positive probability beyond any given age, then there is, in any sufficiently productive economy, no limit on how much a government can borrow.

Outline We introduce the economy with incomplete markets in Section 2. In Section 3, as a benchmark, we describe the stationary complete markets allocations for the underlying economy. Section 4 shows when unbounded deficits are feasible, and when they are desirable. We specialize to a unit elasticity of substitution in Section 5 and provide explicit conditions under which the government can actually run a permanent primary deficit. In Section 6 we prove the utility approximation result for finitely-lived households. Section 7 concludes with important caveats.

2 The Economy

We only consider balanced growth paths of our economy.

2.1 Demographics and Preferences

There is a flow $\delta > 0$ of newborn consumers. At all times, consumers supply L units of labor inelastically. They die randomly at the rate δ. The population is assumed to be in a steady state, so that there is a unit measure of consumers. Given consumption flows $C_{j,t}$ and information generated by a Brownian motion $Z_{j,t}$, the utility process $U_{j,t}$ of a typical consumer j evolves according to

$$\mathrm{d}U_{j,t} = U_{j,t}\left(\mathcal{A}_t U_j \mathrm{d}t + \mathcal{S}_t U_j' \mathrm{d}Z_{j,t}\right)$$

where $\mathcal{A}_t U_j$ and $\mathcal{S}_t U_j$ satisfy

$$(\rho+\delta)U_{j,t}^{1-1/\varepsilon} = (\rho+\delta)C_{j,t}^{1-1/\varepsilon} + \left(1-\frac{1}{\varepsilon}\right)U_{j,t}^{1-1/\varepsilon}\left(\mathcal{A}_t U_j - \frac{1}{2}\xi\|\mathcal{S}_t U_j\|^2\right).$$

This is a continuous-time version of Epstein and Zin [1989] that makes utility homogeneous of degree 1 in consumption (see Duffie and Epstein [1992]). The parameter $\varepsilon \in (0,\infty)$ is the intertemporal elasticity of substitution, and ξ is the coefficient of relative risk aversion. In the standard additively separable case, $\varepsilon = 1/\xi$. Our results apply to this special case, but, following Bansal and Yaron [2004], we are especially interested in scenarios with both ξ significantly above 1 and with $\varepsilon \in (1,\infty)$.

2.2 Idiosyncratic Capital Accumulation

There is a single type of capital in this economy, but consumers can only accumulate this capital subject to idiosyncratic shocks. The capital stock of a consumer j who holds $K_{j,t}$ units of capital, evolves according to

$$\mathrm{d}K_{j,t} = \left(\mu K_{j,t} - X_{j,t}\right)\mathrm{d}t + \varsigma K_{j,t}\mathrm{d}Z_{j,t} + \mathrm{d}I_{j,t},$$

where $X_{j,t} \geq 0$ is a flow of capital used for consumption, $I_{j,t}$ represents cumulative purchases of capital, and $Z_{j,t}$ is a standard Brownian motion that is unrelated across consumers. The parameters μ and $\varsigma > 0$ are common. A central feature of our economy is that there are no financial markets contingent on the $Z_{j,t}$.

2.3 The Aggregate Technology

The technology in the consumption sector is Cobb-Douglas,

$$Y_t = X_t^{1-\alpha} L^{\alpha},$$

where $L > 0$ is inelastically supplied labor, and $X_t \geq 0$ is the flow of capital used up during the process of producing consumption. Throughout, it is assumed that $\alpha \in (0,1)$.[11]

Because idiosyncratic shocks average out, the aggregate capital stock evolves according to

$$\mathrm{d}K_t = \mu K_t \mathrm{d}t - X_t \mathrm{d}t.$$

The price of capital in units of consumption is q_t. Since X_t depletes capital at a one-for-one rate, this is also the factor price of capital faced by producers of consumption. The factor price of labor is w_t in units of consumption. Profit maximization in the consumption sector then implies

$$\begin{bmatrix} q_t X_t \\ w_t L \end{bmatrix} = \begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix} Y_t.$$

A closely related interpretation is that $X_t/\mu \in [0, K_t]$ is capital that is employed to produce consumption rather than new capital. This makes this economy a special case of Uzawa [1965], with a linear technology in the capital accumulation sector. We will refer to this as the Uzawa-AK technology.

[11] Any economy will have to have land, possibly another fixed factor. This is not innocuous, because a claim to labor is not a claim to an infinitely-lived asset, and land could be (Muller and Woodford [1988]). We assume there is an unbounded supply of unimproved land and interpret capital as including improved land.

2.4 Balanced Growth

Suppose $X_t/K_t = x \in (0, \infty)$. Then the aggregate capital stock grows at the rate $\mu - x$, and therefore Y_t grows at the rate

$$g = (1 - \alpha)(\mu - x). \tag{1}$$

This relation describes an immediate trade-off between the level and the growth rate of consumption. Since Y_t grows at the rate $(1-\alpha)(\mu - x)$, and X_t grows at the rate $\mu - x$, the fact that factor shares are constant implies that $(\mathrm{d}q_t/\mathrm{d}t)/q_t = \mu_q$ is given by $\mu_q = -\alpha(\mu - x)$, and therefore $(1-\alpha)\mu_q = -\alpha g$. The technology of this economy says that a high growth rate must go together with a rapidly declining price of capital.[12] This is the same as saying that the aggregate return on capital is given by $\mu + \mu_q = (1-\alpha)\mu + \alpha x$. Using (1), yet another way to put this is

$$x = \mu + \mu_q - g. \tag{2}$$

Notice that the dividend-price ratio for the aggregate capital stock is simply $(q_t X_t)/(q_t K_t) = x$. Therefore, in Gordon-growth fashion, $x = \mu + \mu_q - g$ can be interpreted as the effective discount rate for the dividends produced by the aggregate capital stock.

Let r be the risk-free rate in this economy. It will be useful to note that the right-hand side of (2) can be written as the sum of the excess return $\mu + \mu_q - r$ on capital and the effective discount rate $r - g$ for risk-free dividends that grow at the same rate g as Y_t.

2.5 Government

Household consumption at time t is C_t, and wealth is W_t. The government consumes G_t. The aggregate resource constraint is $C_t + G_t = Y_t$. The target for government consumption is $G_t = \gamma C_t$. The government also targets a consumption tax $\tau \in (-1, \infty)$ and a wealth tax $\omega \in (-\infty, \infty)$. This implies government revenues equal to $T_t = \tau C_t + \omega W_t$. In addition, the government targets aggregate transfers to newborn consumers equal to σY_t. We only consider steady state equilibria that are consistent with the fiscal target parameters γ, τ, σ, and ω. We do not specify government policy for off-the-equilibrium prices that are not consistent with these targets.

[12] This also means that GDP grows faster than the value of aggregate output in units of consumption.

2.5.1 The Primary Surplus or Deficit

Including the consumption tax, household consumption expenditures are $E_t = (1+\tau)C_t$. The primary surplus of the government relative to household consumption expenditures is therefore given by the surplus ratio

$$\mathcal{S}_t = \frac{T_t - G_t - \sigma Y_t}{E_t} = 1 + \frac{\omega}{E_t/W_t} - \frac{(1+\gamma)(1+\sigma)}{1+\tau}. \tag{3}$$

In a steady state, aggregate consumption and wealth grow at the rate g. Write $\mathcal{S}$ for the steady state surplus ratio, and $[C_t, G_t, Y_t, E_t, W_t] = [C, G, Y, E, W]\, e^{gt}$ for the balanced growth path of the consumption sector.

2.5.2 Government Issued Deposits

The government runs a bank that issues deposits. At time $t = 0$, the supply of these deposits is $D_0 > 0$. Purchases of consumption goods are paid for and basic income transfers are made by issuing more deposits. Taxes are used to retire deposits. The government also pays interest on these deposits, by issuing more deposits. For simplicity, take the nominal interest rate to be constant at some real number $i \geq 0$. The price of consumption in units of government deposits is P_t.

If the $1/P_t$ are positive, then the supply D_t of government deposits evolves according to

$$\mathrm{d}D_t = iD_t\mathrm{d}t + P_t(G_t + \sigma Y_t - T_t)\mathrm{d}t,$$

starting from $D_0 > 0$. Since government deposits are risk-free, it must be that $i = r + (\mathrm{d}P_t/\mathrm{d}t)/P_t$. This implies

$$\mathrm{d}\left(\frac{D_t}{P_t E_t}\right) = (r-g)\left(\frac{D_t}{P_t E_t}\right)\mathrm{d}t - \mathcal{S}_t\mathrm{d}t. \tag{4}$$

In a steady state, the surplus ratio (3) will be constant, and so $D_t/(P_t E_t)$ must be constant. Since $E_t = Ee^{gt}$, this means that $D_t/P_t = (D/P)e^{gt}$, where $[D, P] = [D_0, P_0]$. Since the price level grows at the rate $i - r$, this gives $D_t = De^{(i-(r-g))t}$.

Government policy is never to lend to the public. The equilibrium value of government deposits can therefore be zero or positive, but not negative. The steady state value of $D/(PE)$ must therefore satisfy

$$(r-g) \times \frac{D}{PE} = 1 + \frac{\omega}{E/W} - \frac{(1+\gamma)(1+\sigma)}{1+\tau}, \qquad \frac{D}{PE} \geq 0. \tag{5}$$

If the government's budget is balanced, then (5) will hold if $r-g=0$ and also if $r-g\neq 0$ and $D/(PE)=0$. Our assumption that $D_0>0$ means that $D/(PE)=0$ forces $1/P=0$. The fiscal targets γ, τ, σ and ω, together with a government policy not to lend to the public, force $r-g>0$ if these fiscal targets imply a primary surplus, and $r-g<0$ if these fiscal targets imply a primary deficit.

2.6 Aggregate Wealth and Portfolio Shares

Consumers can pledge assets held at their time of death in exchange for an annuity income, as in Yaari [1965] and Blanchard [1985]. Conditional on survival, their labor incomes w_tL grow at the same rate g as aggregate consumption. At birth, date-t newborn consumers sell their future labor income for $w_tL/(\delta+r-g)$ and buy capital and risk-free securities.[13] In any equilibrium, this present value must be well defined and finite. As long as $\alpha\in(0,1)$, this requires $\delta+r-g>0$. Aggregate household wealth at any point in time can then be defined as

$$W_t = q_tK_t + \frac{w_tL}{\delta+r-g} + \frac{D_t}{P_t}.$$

As already anticipated, the steady state conditions imply that $W_t = We^{gt}$.

Let $\psi = qK/W$ be the steady state portfolio share of capital. Given $X=xK$, $qX=(1-\alpha)Y$, and $Y_t/E_t=(1+\gamma)/(1+\tau)$, this implies

$$\psi = \frac{1-\alpha}{x}\frac{1+\gamma}{1+\tau}\frac{E}{W}, \tag{6}$$

where x is given by (2). This can be read as saying that wealth held in the form of capital, ψW, is equal to the present value of the capital income flows $(1-\alpha)Ye^{gt}$, discounted at the effective rate $x=\mu+\mu_q-g$. Using $w_tL=\alpha Y_t$ and $Y_t/E_t=(1+\gamma)/(1+\tau)$, the definition of wealth then implies a risk-free portfolio share

$$1-\psi = \left(\frac{\alpha}{\delta+r-g}\frac{1+\gamma}{1+\tau}+\frac{D}{PE}\right)\frac{E}{W}. \tag{7}$$

One could imagine an economy in which consumers follow inelastic decision rules defined simply by taking E/W and ψ as parameters. Then one can view (6) and (7) as

[13] On a given equilibrium path, this is equivalent to the assumption that consumers can only issue real debt, subject to a present value borrowing constraint. In contrast, the government issues deposits, and this enables it to effectively run a Ponzi scheme when $r-g<0$. One possible interpretation is that the private sector faces legal restrictions against such Ponzi schemes.

market clearing conditions for capital and risk-free assets, respectively. Together with the steady state condition (5) for government deposits, these conditions must be solved for x, $r-g$, and $D/(PE)$. The growth rate of the economy then follows from $g=(1-\alpha)(\mu-x)$.

2.7 Consumer Decision Rules and Utility

With the economy on a balanced growth path, consumers face constant expected rates of return. Although shocks are idiosyncratic, everyone faces the same return parameters.[14] Preferences are homothetic, and so we can take E/W and ψ to also represent the decision rules of individual consumers. Conditional on survival, the average growth rate of individual consumption is given by

$$g_y = r + \delta + \psi(\mu + \mu_q - r) - \left(\omega + \frac{E}{W}\right). \tag{8}$$

Including annuity payments, consumers face expected returns $r+\delta$ and $\mu+\mu_q+\delta$, but the wealth tax lowers the rate at which wealth grows by ω. The Epstein-Zin preferences we have adopted imply

$$\begin{aligned} \frac{E}{W} &= \rho + \delta - \left(1 - \frac{1}{\varepsilon}\right)\left(g_y - \frac{1}{2}\xi\varsigma^2\psi^2\right) && (9) \\ \psi &= \frac{\mu + \mu_q - r}{\xi\varsigma^2}. && (10) \end{aligned}$$

Using (10) to eliminate $\mu+\mu_q-r$ from (8) gives $g_y = r+\delta+\xi\varsigma^2\psi^2-(\omega+E/W)$, and plugging this into (9) gives E/W in terms of r and ψ. The decision rules (8)-(10) are well defined if and only if E/W is strictly positive. The resulting utility for a consumer j with wealth $W_{j,t}$ is determined by

$$U_{j,t} = C_{j,t}\left(\frac{E/W}{\rho+\delta}\right)^{-1/(1-1/\varepsilon)}, \qquad C_{j,t} = \frac{W_{j,t}}{1+\tau}\frac{E}{W}. \tag{11}$$

So $U_{j,t}$ is linear in $W_{j,t}$, with a slope that scales with $(E/W)^{1/(1-\varepsilon)}$. Holding fixed $W_{j,t}$, the partial effects on utility of an increase in the risk-free rate r and of an increase in the risk premium $\mu+\mu_q-r$ are both positive.

The expression (9) for E/W highlights how the consumption-wealth ratio of a consumer whose consumption process follows a geometric Brownian motion depends on

[14] As in Angeletos [2006], the combination of a tradable source of labor income together with an idiosyncratic capital accumulation technology leads to a Merton problem.

its geometric drift parameter g_y, its diffusion coefficient $\varsigma\psi$, and the risk-aversion coefficient ξ. What matters is the risk-adjusted growth rate $g_y - \frac{1}{2}\xi\varsigma^2\psi^2$. When $\varepsilon \in (0,1)$, the consumption-wealth ratio E/W is positive if and only if this risk-adjusted growth rate is not too low. When $\varepsilon \in (1,\infty)$, it is positive if and only if the risk-adjusted growth rate is not too high.

2.8 Equilibrium

Balanced growth paths for this economy can be constructed by solving the market clearing conditions (6)-(7), taking into account the decision rules (8)-(10), as well as (1)-(2) and (5). Note that (2) and (10) immediately imply $x = \xi\varsigma^2\psi + r - g$, and this can be used to eliminate x from (6). That leaves four equilibrium conditions, (5)-(9), that have to be solved for $r - g$, ψ, E/W, and $D/(PE)$.

Observe that $r - g$ and ψ pin down E/W via (8)-(10), and then (7) can be used to infer $D/(PE)$. So $r - g$ and ψ are sufficient to identify a particular equilibrium.

2.8.1 Unforeseen and Permanent Policy Changes

As in AK economies without a fixed factor, our economy will have a balanced growth path from the start, and following unforeseen and permanent changes in government policy. Our assumption that newborn consumers can sell their labor income at birth means that all consumers alive at a point in time use the same portfolio shares for risky and risk-free securities. To predict what happens following an unforeseen change in policy, it is easiest to assume that everyone alive holds physical capital subject to idiosyncratic shocks, as well as shares in a risk-free mutual fund that is backed by labor income and government securities. The value of this mutual fund will typically jump following an unforeseen change in government policy.[15] For all consumers already alive, utilities change one for one with the utility U_t of a consumer whose wealth is equal to aggregate wealth. Importantly, however, all future generations hold unbalanced portfolios, consisting only of claims to their future labor income. The utility of a generation born $T \geq 0$ units of time into the future can be written as $U_{y,t}e^{gT}$. The welfare consequences of an unforeseen and permanent policy change at time t can be accounted for using U_t, $U_{y,t}$, and g.

[15] Because of our assumption that $D_0 > 0$, the mutual fund of risk-free securities will always hold government securities, even if $1/P = 0$. Starting from a $1/P = 0$ scenario, the wealth effects of an unforeseen policy change (or an unforeseen jump between different equilibria) that suddenly gives government securities a positive price will then accrue to all consumers alive in proportion to their wealth.

2.8.2 The Dynamics of Consumption, Wealth and Utility

Conditional on survival, the individual wealth of consumer j evolves according to

$$\mathrm{d}W_{j,t} = W_{j,t}\left(g_y \mathrm{d}t + \psi\varsigma \mathrm{d}Z_{j,t}\right),$$

The wealth distribution is the determined by how g_y differs from the growth rate g of aggregate consumption, and by ψ. Since an unforeseen change in government policy does not redistribute wealth among consumers alive at the time of the change, the distribution of wealth will only adjust to the new policy over time. As pointed out by Gabaix, Lasry, Lions and Moll [2017], this may take a long time.

2.8.3 Aggregate Versus Individual Consumption Growth

The distribution of wealth is driven by idiosyncratic capital accumulation shocks, and by the discrepancy between the drift g_y of individual consumption and the aggregate consumption growth rate g. Newborn consumers start with wealth $W_{y,t} = W_y e^{gt}$, where W_y/W satisfies

$$\sigma \times \frac{1+\gamma}{1+\tau}\frac{E}{W} = \delta\left(\frac{W_y}{W} - \frac{\alpha}{\delta + r - g}\frac{1+\gamma}{1+\tau}\frac{E}{W}\right). \tag{12}$$

The left-hand side gives aggregate transfers σY relative to aggregate wealth, using the fact that $Y/E = (1+\gamma)/(1+\tau)$. The right-hand side accounts for the fact that these transfers add an instantaneous jump $\sigma Y/\delta$ to the initial wealth of every one of these consumers. Observe that (11) and (12) allow one to relate the utility of newborn consumers to the utility of the average consumer.

As a matter of accounting, the ratio W_y/W pins down the relation between aggregate consumption and the drift of individual consumption,

$$g = g_y - \delta\left(1 - \frac{W_y}{W}\right). \tag{13}$$

The term g_y is the consumption growth rate of surviving consumers. The consumption of consumers who randomly die scales with W, while the consumption of newborn consumers is proportional to W_y. Since $W_y > 0$, the accounting equation (13) implies $g_y < g + \delta$. The consumption growth rate of surviving consumers can, on average, exceed the aggregate growth rate of consumption, but by no more than δ.

It should be emphasized that the accounting relation (13) is already implied by (12) and the equilibrium conditions (1)-(2), (5), (6)-(7), and (8)-(10). This is a consequence of Walras' law.

2.8.4 The Stationary Wealth Distribution

By Ito's lemma, we have

$$\mathrm{d}\ln\left(\frac{W_{j,t+a}}{W_{y,t+a}}\right)=\left(g_y-\left(g+\frac{1}{2}\psi^2\varsigma^2\right)\right)\mathrm{d}a+\psi\varsigma\mathrm{d}Z_{j,t+a}.$$

Taking logs reduces the drift g_y by the Ito term $\psi^2\varsigma^2/2$, and de-trending with newborn wealth reduces the drift by g. In a cohort of age a, the distribution of wealth relative to current newborn wealth among surviving consumers is therefore normal with mean $\left(g_y-\left(g+\frac{1}{2}\psi^2\varsigma^2\right)\right)a$ and variance $\psi^2\varsigma^2 a$. Random death implies that the age distribution is exponential with a density $\delta e^{-\delta a}$. Combining these two distributions gives the distribution of wealth relative to newborn wealth. As is well known, this implies a double Pareto distribution.[16]

Proposition 1 *The stationary distribution of wealth relative to newborn wealth, $u_{j,t}=W_{j,t}/W_{y,t}$, has a density given by*

$$f(u)=\frac{\min\left\{u^{-(1+\zeta_-)},u^{-(1+\zeta_+)}\right\}}{\frac{1}{\zeta_-}+\frac{1}{\zeta_+}},\qquad u\in(0,\infty),$$

where

$$\zeta_\pm=-\frac{d}{s^2}\pm\sqrt{\left(\frac{d}{s^2}\right)^2+\frac{\delta}{s^2/2}},\qquad d=g_y-\left(g+\frac{1}{2}\psi^2\varsigma^2\right),\qquad s=\psi\varsigma.$$

This satisfies $\zeta_-<0<\zeta_+$ and $\zeta_+>1$.

The fact that $g_y<g+\delta$ implies that $d+\frac{1}{2}s^2<\delta$, and therefore $\zeta_+>1$. The construction of a balanced growth path therefore guarantees that the distribution of wealth has a finite mean. The closer g_y is to its upper bound $g+\delta$, the closer is the tail index ζ_+ to 1, which is Zipf's law. In turn, this happens when W_y/W is particularly small.[17]

2.9 Baby Bonds versus a Universal Basic Income

As in settings in which classical Ricardian results apply, there is, in our incomplete markets economy, a certain arbitrariness to how government transfers to consumers are implemented.

[16]Because wealth is de-trended by newborn wealth, this proposition generalizes easily to a setting with Brownian aggregate shocks to the technology for accumulating capital.

[17]If $\varepsilon=1$, $\rho\downarrow 0$, and $\xi\varsigma^2\downarrow 0$, and there is no government, then $g_y=g+x$ and $\psi=x/\delta=\sqrt{1-\alpha}$, from (19) below. The resulting wealth distribution is Pareto with a tail index $\zeta_+=\delta/(g_y-g)=1/\sqrt{1-\alpha}$. At $\alpha=5/9$, this gives $\zeta_+=1.5$, in line with US data (see Aoki and Nirei [2017]).

To illustrate, suppose that the government not only makes one-time transfers to newborn consumers but also pays everyone a universal basic income (UBI). Suppose the aggregate UBI transfers are θY_t. The primary surplus of the government is then given by

$$\frac{T_t - G_t - (\sigma + \theta)Y_t}{E_t} = 1 + \frac{\omega}{E_t/W_t} - \frac{(1+\gamma)(1+\sigma+\theta)}{1+\tau},$$

and this appears in the steady state condition (5) for $D/(PE)$. Consumers now receive a universal basic income θY_t along with their labor income $w_t L = \alpha Y_t$. To account for this additional income, the labor share parameter α in (7) and (12) (but not in (6)) must be replaced by $\alpha + \theta$. The remaining equilibrium conditions for $r - g$, ψ, E/W, and $D/(PE)$ are unaffected.

Proposition 2 shows that any policy with a UBI component can be transformed into a policy with only baby bonds, with no effect on the consumption allocation, of anyone, at any time.

Proposition 2 *Suppose the equilibrium for a policy (θ, σ) is given by ψ and $r - g$. Then ψ and $r - g$ are also an equilibrium for the policy (θ', σ') defined by*

$$\theta' = 0, \qquad \sigma' = \sigma + \delta \times \frac{\theta}{\delta + r - g}.$$

Given an unforeseen one-time change in policy from (θ, σ) to (θ', σ'), the price level is also not affected if the government makes an instantaneous transfer of deposits equal to

$$\frac{D' - D}{P} = \frac{\theta Y}{\delta + r - g}$$

to consumers alive at the time of the policy change.

This policy is constructed to leave newborn consumers with the same amount of wealth when the universal basic income is abolished and replaced by baby bonds. And consumers already alive are compensated for the universal basic income they lose as a result of the new policy. This instantaneous transfer causes a jump $D' - D > 0$ in the supply of government deposits. Observe that the definition of σ' implies that the primary surplus of the government changes by the amount

$$-(\mathcal{S}' - \mathcal{S})E_t = -\sigma' Y_t + (\sigma + \theta)Y_t = (r - g) \times \frac{\theta Y_t}{\delta + r - g} = (r - g) \times \frac{D' - D}{P}.$$

If $r - g > 0$, then this implies $\mathcal{S}' < \mathcal{S} \leq 1$, and an increase in the primary surplus of the government. But if $r - g < 0$, then this implies $1 \leq \mathcal{S} < \mathcal{S}'$, and an increase in the primary deficit of the government. In both cases, this implies an increase in the steady state supply of government deposits. The instantaneous transfer of deposits to consumers already alive exactly matches this steady state increase, without a change in the price level. If consumers already alive were not made good by the government for losing their universal basic income, then there would be a drop in the price level that ensures $(r - g)(D/P' - D/P) = -(\mathcal{S}' - \mathcal{S})E_t$. That would give these consumers a capital gain that exactly compensates them for losing the universal basic income.

For the government, the flow cost of making instantaneous transfers with a present value $1/(\delta + r - g)$ to a flow δ of agents is $\delta/(\delta + r - g)$. The flow cost of transferring a unit flow of consumption to a unit measure of agents is 1.[18] Hence, switching from baby bonds to an equivalent UBI could turn a surplus into a deficit if $r - g > 0$, and a deficit into a surplus if $r - g < 0$. In both cases, this would then require the government to lend to the public. Therefore, there is a range of baby bond policies that cannot be replicated using a universal basic income when the government does not lend to consumers.

3 Complete Markets Economies

To set the stage, it is useful to discuss in some detail the effects of alternative fiscal policies on growth and welfare in the complete markets version of our economy.

3.1 Infinitely Lived Consumers

Suppose $\delta = 0$ and that markets are complete. So there is a fixed population of consumers who live forever, and these consumers can perfectly share the idiosyncratic risk of their capital accumulation technologies. This implies a representative consumer. Accounting for possible wealth taxes, the standard Euler condition is then

$$g = \varepsilon(r - (\rho + \omega)). \tag{14}$$

Since there can be no risk premium, $\mu + \mu_q - r = 0$, and hence (2) becomes $x = r - g$. Together with $g = (1 - \alpha)(\mu - x)$, this yields

$$\alpha g = (1 - \alpha)(\mu - r). \tag{15}$$

[18]If $r - g > 0$, then the present value of transfers made to all consumers born after a given initial date is $\delta/((r - g)(\delta + r - g))$ in both cases.

The negative technological relation between the return on capital and the growth rate of the economy becomes a negative relation between the risk-free rate and the growth rate of the economy.

Figure 1 shows the equilibrium (14) and (15). Variation in the productivity μ of the capital accumulation technology results in shifts of (15) along the Euler condition (14), and this leads to co-movement of r and g. On the other hand, variation in ρ produces movements in the Euler condition along the technological restriction (15) on r and g. As long as the factor share of capital is strictly positive, this leads to r and g that move in opposite directions.

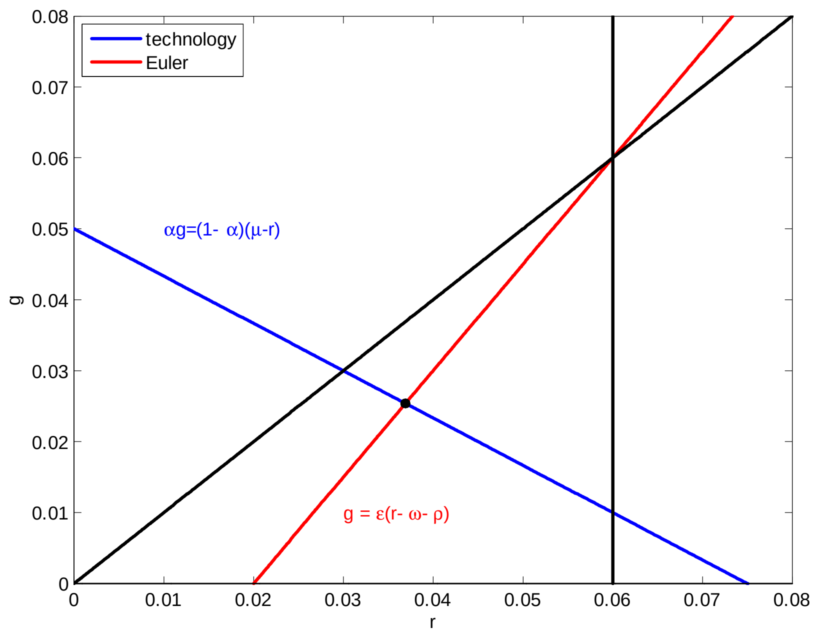

Figure 1 *Equilibrium in the representative agent economy.*

In any equilibrium x, and hence $r - g$, must be positive. In Figure 1, the intersection of $g = \varepsilon(r - (\rho + \omega))$ with the diagonal defines the boundary of the region where this is the case. Here, $\varepsilon \in (1, \infty)$, resulting in an upper bound on r and g for which $r - g$ is positive. The economy only has a well defined equilibrium if $\mu - \omega$ is low enough to ensure that the intersection of $\alpha g = (1 - \alpha)(\mu - \omega - (r - \omega))$ and $g = \varepsilon(r - \omega - \rho)$ is below this upper bound. If $\varepsilon \in (0, 1)$, then the requirement that $r - g > 0$ implies a lower bound on $\mu - \omega$. Away from these boundaries, an increase in the wealth tax lowers the growth rate of this economy. The technology implies that this raises the level of current consumption at the same time.

3.2 Perpetual Youth

When there is a flow $\delta > 0$ of newborn consumers and consumers die randomly at some rate $\delta > 0$, the economy no longer has a representative agent. Ricardian equivalence fails and fiscal policy can have important effects on the growth rate of the economy.[19]

Continuing to assume that markets are complete, the fact that $r - g = x > 0$ implies that the present value of aggregate consumption will be finite. If the wealth tax is zero, this will be enough to guarantee that competitive equilibria are Pareto efficient. The standard proof of the first welfare theorem works. The Uzawa-AK technology we are using automatically rules out the dynamic inefficiencies that are central in Samuelson [1958] and Diamond [1965].[20]

3.2.1 The Equilibrium Conditions

The fact that $x = r - g > 0$ in any equilibrium, together with our assumption that the government does not lend to consumers, also implies that the government cannot run permanent primary deficits. For simplicity, focus on the case $(1+\gamma)(1+\sigma)/(1+\tau) \leq 1$, so that wealth taxes are never necessary to avoid a deficit.

Eliminating ψ from the market clearing conditions (6)-(7) and using (5) gives

$$1 = \left(\frac{1-\alpha}{x} + \frac{\alpha}{\delta + x}\right)\frac{1+\gamma}{1+\tau}\frac{E}{W} + \frac{1}{x}\left(\omega + \left(1 - \frac{(1+\gamma)(1+\sigma)}{1+\tau}\right)\frac{E}{W}\right). \tag{16}$$

This is simply a decomposition of the components of consumer wealth, into capital, claims to labor, and government securities. The absence of idiosyncratic risk means that the consumer decision rules (8)-(10) reduce to

$$g = (1-\alpha)(\mu - x), \qquad g_y = g + \delta + x - \left(\omega + \frac{E}{W}\right), \qquad \frac{E}{W} = \rho + \delta - \left(1 - \frac{1}{\varepsilon}\right) g_y. \tag{17}$$

The side conditions are $x > 0$ and $E/W > 0$.

Using the fact that $\delta > 0$, $\sigma \geq 0$, and $\omega \geq 0$, it is easy to see from (16) that $x \in (\omega, \omega + E/W)$ in any equilibrium. The second condition in (17) then ensures that $g_y < g + \delta$ in any equilibrium. Note that (17) and $x = r - g$ implies the Euler condition $g_y = \varepsilon(r - (\rho + \omega))$ for individual consumption growth. The Euler condition (14) for aggregate consumption growth no longer applies.

[19] Detailed proofs for Section 3.2 are provided in the online appendix.

[20] It is easy to show that this is still true when the Brownian motions $Z_{j,t}$ that govern the individual capital accumulation technologies are correlated, which introduces aggregate uncertainty.

Given a solution to these equilibrium conditions, and an initial capital stock K, the utility U of the average consumer already alive and the utility U_y of the current generation of newborn consumers are

$$U = \frac{(xK)^{1-\alpha}\, L^{\alpha}}{1+\gamma}\left(\frac{E/W}{\rho+\delta}\right)^{-1/(1-1/\varepsilon)}, \qquad U_y = \frac{g+\delta-g_y}{\delta}\times U. \tag{18}$$

This follows from (11) and the fact that $W_y/W = (g+\delta-g_y)/\delta$ in any steady state. Since aggregate consumption grows at the rate g, the utility of a consumer who will be born at some future date $T \geq 0$ is $U_y e^{gT}$.

3.2.2 Existence of Equilibrium

Solving both (16) and (17) for E/W and eliminating g and g_y gives

$$\frac{E}{W} = \frac{1-\frac{\omega}{x}}{\left(\frac{1-\alpha}{x}+\frac{\alpha}{\delta+x}\right)\frac{1+\gamma}{1+\tau}+\frac{1}{x}\left(1-\frac{(1+\gamma)(1+\sigma)}{1+\tau}\right)}, \tag{19}$$

$$\frac{E}{W} = \varepsilon\times\left(\rho+\delta-\left(1-\frac{1}{\varepsilon}\right)((1-\alpha)(\mu-\omega)+\alpha(x-\omega)+\delta)\right). \tag{20}$$

By taking derivatives, one can verify that (19) is a positive, increasing, and concave function of $x \in (\omega, \infty)$. Importantly, its slope converges to a limit that is greater than or equal to 1 as x becomes large. The equation (20) is a line with slope $(1-\varepsilon)\alpha < 1$. Therefore, the only way these two equilibrium conditions can intersect with $E/W > 0$ is for the line to be strictly positive at $x = \omega$.

As in the economy with infinitely lived consumers, an increase in the wealth tax ω implies an increase in x, because (19) implies $\partial(E/W)/\partial\omega = -(E/W)/(x-\omega) < -1$ while (20) gives $\partial(E/W)/\partial\omega = -(1-\varepsilon) > -1$. This increase in x lowers the growth rate $g = (1-\alpha)(\mu-x)$ and increases the risk-free rate $r = \alpha x + (1-\alpha)\mu$. Increases in τ and reductions in $\sigma > 0$ shift (19) down, which also raises x. This proves our next proposition.

Proposition 3 *Suppose the fiscal targets of the government satisfy $(1+\gamma)(1+\sigma)/(1+\tau) \leq 1$, and that there is a wealth tax $\omega \geq 0$. Then the complete markets economy has an equilibrium if and only if*

$$\left(1-\frac{1}{\varepsilon}\right)\frac{(1-\alpha)(\mu-\omega)+\delta}{\rho+\delta} < 1. \tag{21}$$

The equilibrium is unique and satisfies $x \in (\omega, \omega+E/W)$. Increases in ω and τ, and reductions in $\sigma > 0$, all increase the level of aggregate consumption and lower its growth rate. Budget-neutral increases in σ and τ are good for growth.

Roughly speaking, permanent primary surpluses drive up interest rates and are bad for growth. The key mechanism is seen most easily by considering the special case $\varepsilon = 1$, so that x is determined by $E/W = \rho + \delta$ and (19). It is then immediate that larger surpluses require a higher effective discount rate x, which lowers growth. Setting $\delta = 0$ and $\sigma = 0$ in (19) confirms that this is a crowding out effect that is not present in an economy with infinitely lived consumers.

The bound (21) holds trivially if $\varepsilon = 1$. For $\varepsilon \in (0, 1)$, this bound is a lower bound on $(1 - \alpha)(\mu - \omega) + \delta$, while for $\varepsilon \in (1, \infty)$ it is an upper bound. In any stationary allocation, the deterministic rate at which individual consumption can grow must satisfy $g_y < g + \delta$, and a planner can take $g = (1-\alpha)(\mu-x)$ up to $(1-\alpha)\mu$ by taking x close to zero. Therefore, evaluated at $\omega = 0$, (21) is simply the bound that $(1 - \alpha)\mu + \delta$ must satisfy for the utility of every stationary allocation to be positive if $\varepsilon \in (0, 1)$, and finite if $\varepsilon \in (1, \infty)$.[21]

More generally, $x > \omega$ together with $g_y < g+\delta$ implies that the rate at which individual consumption can grow in a competitive equilibrium has to be less than $(1 - \alpha)(\mu - \omega) + \delta$. As a consequence, an increase in the wealth tax ω shrinks the set of economies with a well-defined complete markets equilibrium if $\varepsilon \in (0, 1)$, and expands it when $\varepsilon \in (1, \infty)$. In particular, when $\varepsilon \in (1, \infty)$, a large enough ω will ensure that (21) holds.[22]

3.2.3 Unbounded Utilities

Suppose $\varepsilon \in (1, \infty)$ and (21) is violated at $\omega = 0$. Then there will be a $\omega_\infty > 0$ so that the economy has a well defined complete markets equilibrium for all $\omega > \omega_\infty$. This ω_∞ is obtained by forcing (21) to hold with equality at $\omega = \omega_\infty$. Write the right-hand side of (20) in terms of this ω_∞ and then eliminate E/W from (19)-(20). The resulting equilibrium condition for x implies $(\varepsilon - 1)\alpha(x - \omega) < (\varepsilon - 1)(1 - \alpha)(\omega - \omega_\infty)$. Since the equilibrium satisfies $x - \omega > 0$, it follows that $x - \omega \downarrow 0$ as $\omega \downarrow \omega_\infty$. It is not difficult to strengthen this to show that $(x - \omega)/(\omega - \omega_\infty)$ has a positive and finite limit as $\omega \downarrow \omega_\infty$. In turn, (19) or (20) then also guarantee that $(E/W)/(\omega - \omega_\infty)$ has a finite and positive limit as $\omega \downarrow \omega_\infty$. Finally, the second equation in (17) can be written as $g + \delta - g_y = (E/W) - (x - \omega)$. This implies that $(g + \delta - g_y)/(\omega - \omega_\infty)$ converges as $\omega \downarrow \omega_\infty$, and an explicit calculation proves that this limit is also positive.

Putting these results together with (18) shows that $U/(\omega - \omega_\infty)^{-1/(1-1/\varepsilon)}$ and $U_y\,/(\omega - \omega_\infty)^{1-1/(1-1/\varepsilon)}$ converge to positive and finite limits as $\omega \downarrow \omega_\infty$. So both U and U_y explode as

[21] Figure 3 shows the $\omega = 0$ version of the bound (21) together with incomplete markets bounds reported in Propositions 5 and 7.

[22] If $\varepsilon \in (0, 1)$, then a large enough negative ω will ensure that the inequality (21) holds, even if it fails at $\omega = 0$. But a planner cannot deliver positive utility if (21) fails at $\omega = 0$, and hence there can be no equilibrium either.

$\omega \downarrow \omega_\infty$. If $\varepsilon \in (1, \infty)$ and $\omega_\infty > 0$, the economy is so productive that an omniscient central planner can deliver unbounded utilities to everyone. But the economy does not have a complete markets equilibrium if $\omega = 0$. Competitive assumptions must be abandoned and we can only speculate what will happen if the government follows such a policy. By setting a wealth tax $\omega > \omega_\infty$, the government can ensure the economy does have a complete markets equilibrium. Since lowering $\omega > \omega_\infty$ increases the growth rate of the economy, there are unbounded Pareto improvements as the government lowers this wealth tax towards ω_∞.

3.2.4 A Ricardian Corollary

Consider some $\omega \geq 0$ for which (21) holds, and fix the solution to (16)-(17) obtained for certain fiscal targets γ, τ, and σ. These fiscal targets only appear in (16). One way to rewrite (16) is

$$\left(1 - \frac{x - \omega}{E/W}\right) \frac{1 + \tau}{1 + \gamma} = \sigma + \frac{\delta \alpha}{\delta + x}. \tag{22}$$

This confirms $x \in (\omega, \omega + E/W)$, as argued in Proposition 3. Given the fixed solution for x and E/W, one can choose any $\tau > -1$ and $\sigma \geq 0$ subject to this affine restriction and obtain the same equilibrium. In particular, one could set $\sigma = 0$. But one can also let τ and σ increase without bound. In that case, the first term on the right-hand side of (16) must converge to zero. From (6), this means that ψ converges to zero, and so does the portfolio weight of labor income. The condition (16) immediately implies that the limiting value of $(1 + \gamma)(1 + \sigma)/(1 + \tau)$ is $(x - \omega)/(E/W)$, and this is strictly inside $(0, 1)$.

In other words, for every feasible government policy, there is an unbounded range of equivalent policies, all implying the same equilibrium allocation of consumption, in which the government uses large consumption taxes to make large transfers to newborn consumers and run a primary surplus. Since E/W and the trajectory for aggregate consumption are the same across all these policies, and since $E_t = (1 + \tau)C_t$, the construction of these equivalent policies implies that W_t scales with $1 + \tau$. When taxes and transfers are large, government securities account for almost all consumer wealth. An unforeseen Ricardian increase in τ and σ will cause an upward jump in aggregate wealth that compensates everyone for the higher consumption taxes they will have to pay.

3.2.5 The Welfare Properties of Stationary Allocations

Proposition 3 says that the economy has a unique equilibrium without wealth taxes if (21) holds at $\omega = 0$. The equilibrium generates a stationary allocation, and it is Pareto efficient. Stationary allocations that are not necessarily Pareto efficient are of interest more generally. They can arise when fiscal parameters are decided once and for all at some initial date, and in environments in which allocations can be adjusted over time by a sequence of social planners.[23]

Stationary Allocations Stationary allocations are characterized by pairs (x, g_y) with $x > 0$ and $g_y < g + \delta = (1-\alpha)(\mu - x) + \delta < (1-\alpha)\mu + \delta$. The resource constraint on aggregate consumption C and newborn consumption C_y is $gC = (g_y - \delta)C + \delta C_y$. This implies $C_y/C = (g + \delta - g_y)/\delta$, as in (13). The resulting utilities are

$$U = \frac{(xK)^{1-\alpha}\, L^{\alpha}}{1+\gamma} \left(1 - \left(1 - \frac{1}{\varepsilon}\right) \frac{g_y}{\rho + \delta}\right)^{-1/(1-1/\varepsilon)}, \qquad U_y = \frac{(1-\alpha)(\mu - x) + \delta - g_y}{\delta} \times U.$$

If $\varepsilon \in (0,1)$, then there are pairs (x, g_y) for which $U \in (0, \infty)$ and $U_y/U > 0$ if and only if (21) holds at $\omega = 0$. If $\varepsilon \in (1, \infty)$, such pairs always exist, but (21) at $\omega = 0$ is needed to ensure utility is finite for all such pairs. It is easy to see that U is increasing in x and g_y. Also, U_y has a unique maximizer $x = ((1-\alpha)\mu + \delta - g_y)/(2-\alpha)$ given any g_y that satisfies $(1 - 1/\varepsilon) g_y < \rho + \delta$ and $g_y < (1-\alpha)\mu + \delta$. This is, in fact, the maximizer of C_y given such g_y. The resulting C_y is decreasing in g_y, by the envelope theorem, while U/C is increasing in g_y. One can verify that the newborn utility $U_y = C_y \times U/C$ has a unique global maximum if (21) holds at $\omega = 0$ and $\varepsilon < 1 + 1/(1-\alpha)$. But if ε is larger, then U_y maximized over x is decreasing in g_y. In that case, U_y is unbounded. The negative effect of lowering g_y on U/C is outweighed by the concomitant increase in the level of newborn consumption.

Starting from some $U \in (0, \infty)$ and $U_y \in (0, \infty)$, an increase in x can never be a Pareto improvement, even if it increases U and U_y, because consumers who will be born sufficiently far into the future will be hurt more by the reduction in g. As long as (21) holds at $\omega = 0$, stationary allocations are Pareto efficient if and only if $g_y = \varepsilon(x + (1-\alpha)(\mu - x) - \rho)$ and $g_y < (1-\alpha)(\mu - x) + \delta$.[24] Figure 2 shows the line $g_y = (1-\alpha)(\mu - x) + \delta$ and the indifference curves of U and U_y for the Pareto efficient allocation that maximizes U_y. In this example, $\varepsilon \in (1, 1 + 1/(1-\alpha))$, so that U_y has a global maximum. Also shown are the allocation that maximizes U, the limiting Pareto efficient allocation that maximizes aggregate growth, and the allocation that maximizes both g and g_y.

[23]For example, consider a two-period overlapping generations version of this economy with logarithmic utility in which a sequence of planners maximize a weighted average of the utilities of those alive at a point in time. This leads to a subgame perfect equilibrium in which the allocation is stationary and not Pareto efficient.

[24]The equation for g_y is an Euler condition for the planner. It is easiest to infer this indirectly, taking for granted that competitive equilibria with $\omega = 0$ are efficient. In such equilibria, there will be an Euler condition $g_y = \varepsilon(r - \rho)$, and $r = x + (1-\alpha)(\mu - x)$.

These four allocations generate a convex quadrilateral. The upward sloping diagonal of this quadrilateral represents the Pareto efficient allocations that are stationary. The edge that connects the maximizers of U_y and U corresponds to the contract curve $(\partial U_y/\partial x)/(\partial U_y/\partial g_y) = (\partial U/\partial x)/(\partial U/\partial g_y)$. The edge $g_y = \varepsilon((1-\alpha)(\mu - x) - \rho)$ that connects the maximizer of U_y and the Pareto efficient allocation that maximizes g corresponds to $\partial U_y/\partial g_y = 0$. This approximates the contract curve for the current newborn generation and generations that will be born very far into the future. One can use this to argue that this convex quadrilateral is the set of stationary allocations that are not Pareto dominated by other stationary allocations. Only the allocations on its upward-sloping diagonal are Pareto efficient.

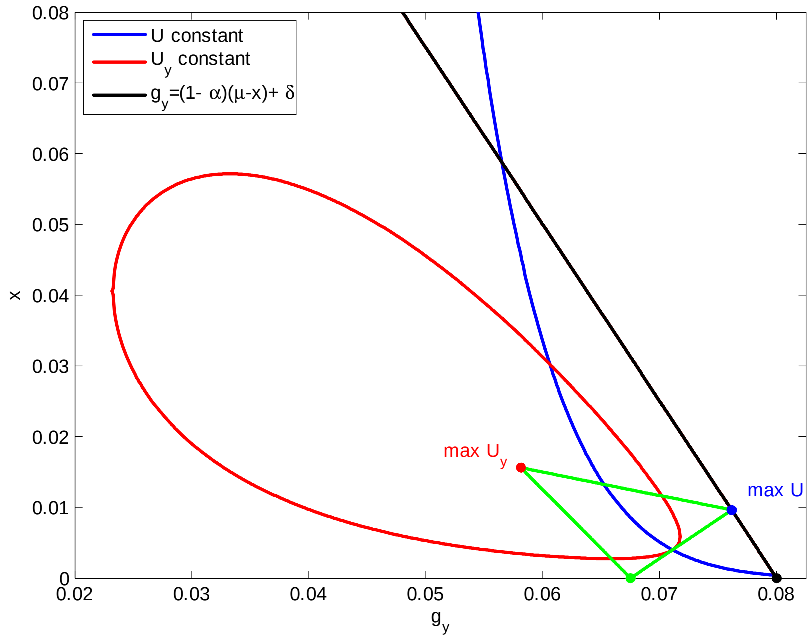

Figure 2 *Stationary allocations with perfect risk sharing*

Implementation The full range of stationary Pareto efficient allocations can be implemented by setting $\omega = 0$ and varying τ and σ. At one end of this range is the allocation preferred by consumers already alive. Maximizing U over x and g_y subject to $g_y = (1-\alpha)(\mu - x) + \delta$ shows that this allocation can be found by imposing $E/W = x$ in (20). Given that we restrict attention to non-negative transfers σ, the implementation of this allocation requires that $(1+\gamma)(1+\sigma)/(1+\tau) \downarrow 0$, so that (19) also reduces to $E/W = x$. As illustrated by Figure 2, $g_y \uparrow g + \delta$ and therefore $C_y/C \downarrow 0$ in this allocation. Every new generation has to start with a very low initial level of consumption when its growth rate g_y conditional on survival is close to the maximal feasible rate $g + \delta$. Proposition 1 implies that consumption inequality approaches Zipf's law. At the other end of the range of stationary Pareto efficient allocations, approximating a competitive equilibrium with an aggregate growth rate that approaches its technological upper bound $(1-\alpha)\mu$ requires balancing the budget and taking τ and σ to be large. This delivers the allocations preferred by generations that will be born very far into the future.

It is not difficult to verify that the contract curve between consumers already alive and the current generation of newborn consumers (the upper edge of the quadrilateral) is given by $x = \rho + \delta - (1 - 1/\varepsilon)\, g_y$. This amounts to imposing $E/W = x$ in (20), which yields $g_y = (1-\alpha)(\mu - x) + \delta - \omega$. Moving towards the allocation that maximizes U_y along this contract curve increases ω. The equilibrium condition (19) can be used to back out Ricardian combinations of τ and σ, as in (22).

4 Incomplete Markets Economies

In a complete markets economy with perpetual youth, a government that does not lend to the private sector cannot run a permanent primary deficit. When markets are incomplete, this is no longer true. In our setting, the perpetual youth assumption is essential, since the present value of the labor income of an infinitely lived consumer would have to be finite in any equilibrium. That would force $r - g > 0$, even when markets are incomplete.

4.1 A Summary of the Equilibrium Conditions

Recall from (5) that the value of government securities outstanding must satisfy,

$$(r-g) \times \frac{D/P}{W} = \omega + \left(1 - \frac{(1+\gamma)(1+\sigma)}{1+\tau}\right)\frac{E}{W}, \qquad \frac{D/P}{W} \geq 0. \tag{23}$$

The risky and risk-free market clearing conditions (6)-(7) are

$$\psi = \frac{1-\alpha}{x}\frac{1+\gamma}{1+\tau}\frac{E}{W}, \tag{24}$$

$$1-\psi = \frac{\alpha}{\delta + r - g}\frac{1+\gamma}{1+\tau}\frac{E}{W} + \frac{D/P}{W}. \tag{25}$$

From (1)-(2) and (10), both x and g are functions of ψ and $r-g$,

$$x = \xi\varsigma^2\psi + r - g, \qquad g = (1-\alpha)(\mu - x). \tag{26}$$

The consumer decision rules (8)-(9) then become

$$\begin{aligned} g_y &= g + \delta + x - \xi\varsigma^2\psi(1-\psi) - \left(\omega + \frac{E}{W}\right), && (27)\\ \frac{E}{W} &= \rho + \delta - \left(1 - \frac{1}{\varepsilon}\right)\left(g_y - \frac{1}{2}\xi\varsigma^2\psi^2\right). && (28) \end{aligned}$$

The side conditions are $\psi \in (0,1)$, $x > 0$, $E/W > 0$, and $r - g \in (-\delta, \infty)$.

Given a solution to these equilibrium conditions, and an initial capital stock K, the utility U of consumers already alive and U_y of newborn consumers are determined by the same formulas (18) as in the complete markets economy. But E/W includes now the risk adjustment $-\frac{1}{2}\xi\varsigma^2\psi^2$ and $g_y - (g + \delta)$ includes a term $\xi\varsigma^2\psi^2$ that reflects expected returns on wealth in excess of r, as well as a linear term $-\xi\varsigma^2\psi = -(\mu + \mu_q - r)$ that arises because, for given x, more risk lowers $r - g$.

4.1.1 Solving this System

These equilibrium conditions can be reduced to two equations in $\psi = (\mu + \mu_q - r)/(\xi\varsigma^2)$ and $r - g$ that can be interpreted as risky and risk-free market clearing conditions. First, using (26)-(28) to eliminate g_y and g gives x and E/W as functions of ψ and $r - g$. Then, using (26)-(28) to eliminate x and E/W from (24) gives a risky market clearing condition. Given ψ, it is a linear equation in $r - g$. And using (23) and (26)-(28) to eliminate $(D/P)/W$ and E/W from (25) gives a risk-free market clearing condition. This is a quadratic equation in ψ given $r - g \neq 0$.

4.2 Primary Surplus Policies

In the complete markets economy, we know that there is an unbounded range of fiscal targets τ and σ, parameterized by (22), that all implement the same consumption allocation. By (24), these large τ and σ imply a small portfolio share ψ of capital. In the incomplete markets economy, a similar policy does, usefully, affect consumption, by making the exposure of individual consumers to idiosyncratic risk small. This gives rise to the following approximation result.

Proposition 4 *Suppose $\omega \geq 0$ and the economy satisfies the condition (21) for the existence of a complete markets equilibrium when the fiscal parameters satisfy $(1+\gamma)(1+\sigma)/(1+\tau) = \Lambda$ for some $\Lambda \in (0, 1]$. Then the resulting complete markets utilities can be approximated in the incomplete markets economy by taking τ and σ to be large, subject to $(1+\gamma)(1+\sigma)/(1+\tau) \to \Lambda$.*

The ψ and $r - g$ that form part of a complete markets equilibrium will also approximately solve the conditions for an incomplete markets equilibrium, because the terms $\xi\varsigma^2\psi$, $\xi\varsigma^2\psi(1-\psi)$, and $\frac{1}{2}\xi\varsigma^2\psi^2$ in (26)-(28) will be small. Given a large-τ and large-σ complete markets solution for ψ and $r-g$, the continuity of (26)-(28) implies that the resulting E/W must be close to the complete markets solution. To make this an equilibrium in the incomplete markets economy, one can then use (24) to back out a τ, and the combination of (23) and (25) to back out σ.

4.2.1 Unbounded Utilities Again

If $\varepsilon \in (0,1)$, then a violation of (21) at $\omega = 0$ means that μ is so low that there are no stationary allocations that deliver positive utility, not even when these allocations are risk-free. This certainly rules out the existence of competitive equilibria in an economy with incomplete markets.

Violations of (21) at $\omega = 0$ are more interesting when $\varepsilon \in (1,\infty)$. We have already shown, for $\varepsilon \in (1,\infty)$, that the complete markets economy has a unique equilibrium as long as $\omega > \omega_\infty$, where $\omega_\infty > 0$ is the value of ω at which (21) holds with equality. And utilities are unbounded as $\omega > \omega_\infty$ approaches ω_∞. In combination with Proposition 4, this means that the government can also deliver unbounded utilities in the incomplete markets economy. It can set $\omega > \omega_\infty$ close to ω_∞, impose large consumption taxes, and use these taxes to make large transfers to newborn consumers as well as back its outstanding securities.

4.3 Balanced Budget Policies

We restrict attention to economies in which both $\omega = 0$ and $(1+\gamma)(1+\sigma)/(1+\tau) = 1$. The surplus ratio $\mathcal{S}$ would be an equilibrium variable if $\omega > 0$, and that complicates the definition of balanced budget policies.

We begin with a proposition that gives the range of economies for which the incomplete markets economy has a balanced budget equilibrium in which the price of government securities is zero—a no-bubble equilibrium. This benchmark will help us characterize the economies for which unbounded government deficits are possible. The proof is in the appendix.

Proposition 5 *Suppose that $\omega = 0$ and budgets are balanced. The economy has a no-bubble* *equilibrium for every $\sigma \geq 0$ if and only if*

$$\left(1-\frac{1}{\varepsilon}\right)\frac{(1-\alpha)\mu+\delta}{\rho+\delta} < 1+\left(1-\frac{1}{\varepsilon}\right)\frac{\delta}{\rho+\delta}\times\begin{cases}\frac{1}{2}\frac{\xi\varsigma^2}{\delta} & \text{if } \frac{\xi\varsigma^2}{\delta}<1,\\ 1-\frac{1}{2}\frac{1}{\xi\varsigma^2/\delta} & \text{if } \frac{\xi\varsigma^2}{\delta}>1.\end{cases} \tag{29}$$

For $\varepsilon \in (0,1)$, the bound (29) on μ is necessary and sufficient for equilibrium given any particular $\sigma \geq 0$. But for large enough $\varepsilon \in (1,\infty)$ there are economies with μ more productive than (29) that do have an equilibrium for small $\sigma \geq 0$ but not for large $\sigma \geq 0$.

If $\varepsilon \in (0,1)$, then the lower bound (29) on μ is tighter than the complete markets bound (21). The distance between these bounds is continuous, increasing, and bounded in $\xi\varsigma^2/\delta \geq 0$. When consumers are exposed to idiosyncratic risk, the economy has to be more productive in order to ensure positive utility. In contrast, if $\varepsilon \in (1,\infty)$, then the complete markets bound (21) is tighter than the upper bound (29). To ensure finite utility, risk-adjusted consumption growth rates cannot be too high, and this constraint is harder to satisfy when there is no consumption risk.

4.3.1 Pure Bubble Equilibria

Now suppose again that budgets are balanced, but consider equilibria in which the price of government securities is strictly positive. These are equilibria in which the price of government securities is a pure bubble. Such bubbles are ruled out in the complete markets economy. As we will now show, pure bubbles are a possibility in our incomplete markets economy.[25]

The conditions for a pure bubble equilibrium can be obtained from (24)-(28) by setting $r-g=0$ and replacing the risk-free market clearing condition (25) with the inequality $1-\psi > (\alpha/\delta)(E/W)/(1+\sigma)$. This inequality ensures that the value of government securities is strictly positive.

Using the risky market clearing condition (24) to eliminate $(E/W)/(1+\sigma)$ from $1-\psi > (\alpha/\delta)(E/W)/(1+\sigma)$, and noting that $x=\xi\varsigma^2\psi$, gives

$$\psi+\frac{\xi\varsigma^2}{\delta}\frac{\alpha}{1-\alpha}\times\psi^2<1. \tag{30}$$

This inequality will be satisfied for all $\psi \in (0,1)$ close enough to zero, and a large $\xi\varsigma^2/\delta$ forces ψ to be close to zero. Given that $1+\sigma=(1+\tau)/(1+\gamma)$, the risky market clearing condition (24) can be written as $E/W=\psi x\times(1+\sigma)/(1-\alpha)$. Using the decision rule

[25]If such unbacked government securities trade at a positive price, then there are additional equilibria in which other unbacked securities trade at a positive price as well, as in Kareken and Wallace (1981). The equilibrium conditions only pin down the aggregate value of public and private unbacked securities.

(27)-(28) to eliminate E/W from this condition, and then (26) to eliminate x and g from the result, gives

$$\frac{\xi\varsigma^2}{\rho+\delta}\left(\frac{1+\sigma}{\varepsilon}\frac{\psi^2}{1-\alpha}+\left(1-\frac{1}{\varepsilon}\right)\left(\alpha\psi-\frac{1}{2}\left(1-(1-\psi)^2\right)\right)\right)=1-\left(1-\frac{1}{\varepsilon}\right)\frac{(1-\alpha)\mu+\delta}{\rho+\delta}. \tag{31}$$

A portfolio weight $\psi \in (0,1)$ defines a pure bubble equilibrium if and only if it satisfies (30) and (31) for some $\sigma \geq 0$.

The properties of the left-hand side of (31) are key. As long as ψ is strictly positive, this left-hand side is strictly increasing and unbounded in $\sigma \geq 0$. And for any given $\sigma \geq 0$, the left-hand side of (31) is a convex quadratic in ψ, equal to zero at $\psi = 0$, with a positive slope at $\psi = 0$ if $\varepsilon \in (0,1)$, and a negative slope at $\psi = 0$ if $\varepsilon \in (1,\infty)$.

If $\varepsilon \in (0,1)$, then it is now easy to see that (31) can be solved for some $\psi \in (0,1)$ if and only if its right-hand side is positive. The resulting ψ will converge to zero as σ becomes large, and so both (30) and (31) will be satisfied for all $\sigma \geq 0$ large enough.

If $\varepsilon \in (1,\infty)$, then the left-hand side of (31) equals zero at $\psi = 0$ and at some $\psi > 0$. This makes pure bubble equilibria possible even if the right-hand side of (31) is negative but close enough to zero. If that right-hand side is positive, then (31) has precisely one solution for $\psi > 0$, and that solution converges to zero as $\sigma \geq 0$ becomes large. Again, this guarantees a pure bubble equilibrium for all $\sigma \geq 0$ large enough.

Since the right-hand side of (31) is equal to zero precisely when the complete markets bound (21) holds with equality at $\omega = 0$, this proves the following proposition.

Proposition 6 *Fix $\omega = 0$ and $\gamma \geq 0$. If $\varepsilon \in (0,1)$, then the condition (21) is necessary and sufficient for the economy to have pure bubble equilibria for all large enough $\tau \geq 0$ and $\sigma \geq 0$ that satisfy $(1+\gamma)(1+\sigma)/(1+\tau) = 1$. If $\varepsilon \in (1,\infty)$, then the condition (21) is sufficient but not necessary.*

In other words, if the economy has a complete markets equilibrium for $\omega = 0$, then there exist balanced budget policies for which the incomplete markets economy also has a pure bubble equilibrium.

If $\varepsilon \in (0,1)$, then, even at $\sigma = 0$, one can solve (30)-(31) for $\psi \in (0,1)$ if the right-hand side of (31) is positive but close enough to zero. One can further show that bubble equilibria for $\sigma = 0$ continue to exist as μ increases up to the lower bound (29) required for a no-bubble equilibrium. So economies in between the thresholds (21) and (29) have pure bubble equilibria at $\sigma = 0$, even though they do not have a no-bubble equilibrium.

If $\varepsilon \in (1,\infty)$, then the fact that the left-hand side of (31) dips below zero for all ψ close enough to zero implies that the lower bound on μ for which the $\sigma = 0$ economy has a pure bubble equilibrium is at or below the threshold that defines the complete markets upper bound (21) on μ.

4.3.2 Comparing These Bounds

Figure 3 summarizes the results of Propositions 3, 5 and 6. It shows the $\omega = 0$ version of the complete-markets bound (21) together with the bound (29) for a no-bubble equilibrium. For $\varepsilon \in (0, 1)$, these are lower bounds on μ, while for $\varepsilon \in (1, \infty)$, they are upper bounds.

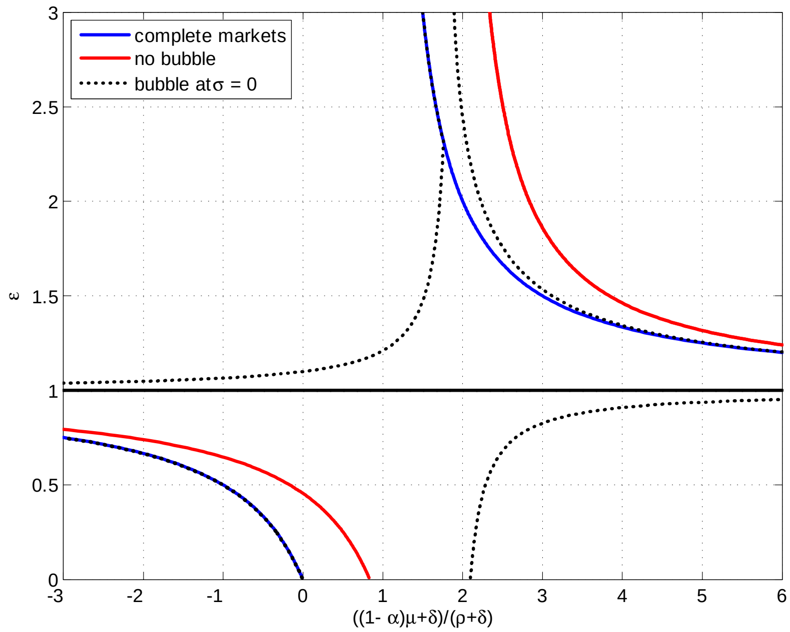

Figure 3 *The complete and incomplete markets bounds*

In the background, the dotted curves in Figure 3 show the upper and lower bounds on μ for which the $\sigma = 0$ economy has a pure bubble equilibrium. Proposition 6 implies that an increase in τ and σ shifts the upper bound to the right if $\varepsilon \in (0, 1)$ and the lower bound to the left if $\varepsilon \in (1, \infty)$. One can also show that the upper bound on μ collapses to the complete markets bound if $\varepsilon \in (1, \infty)$ and σ becomes large. In Figure 3, the region where the $\sigma = 0$ economy has a pure bubble equilibrium includes all economies with $\varepsilon = 1$. This will not be the case when the upper bound for $\varepsilon \in (0, 1)$ and the lower bound for $\varepsilon \in (1, \infty)$ are both upward sloping curves. We will return to this below, where we study the $\varepsilon = 1$ in much more detail.

4.4 Primary Deficit Policies

It can be shown that incomplete markets economies that have a balanced budget equilibrium with a strictly positive bubble also have an equilibrium when the government runs a small permanent primary deficit. This is very similar to what happens in the basic overlapping generations exchange economy with two-period lived consumers. Here we focus on a much more striking result for our incomplete markets economy: there are economies in which there is no bound on how large primary deficits can be.

4.4.1 The Possibility of Unbounded Deficits

To illustrate, Figure 4 shows an example of the risky and risk-free market clearing conditions in an economy with $\varepsilon \in (1, \infty)$. The figure also shows the line $0 = \xi\varsigma^2\psi + r - g$, and a curve that gives the large-σ limit of the risk-free market clearing condition.[26] The risky market clearing condition does not depend on σ.

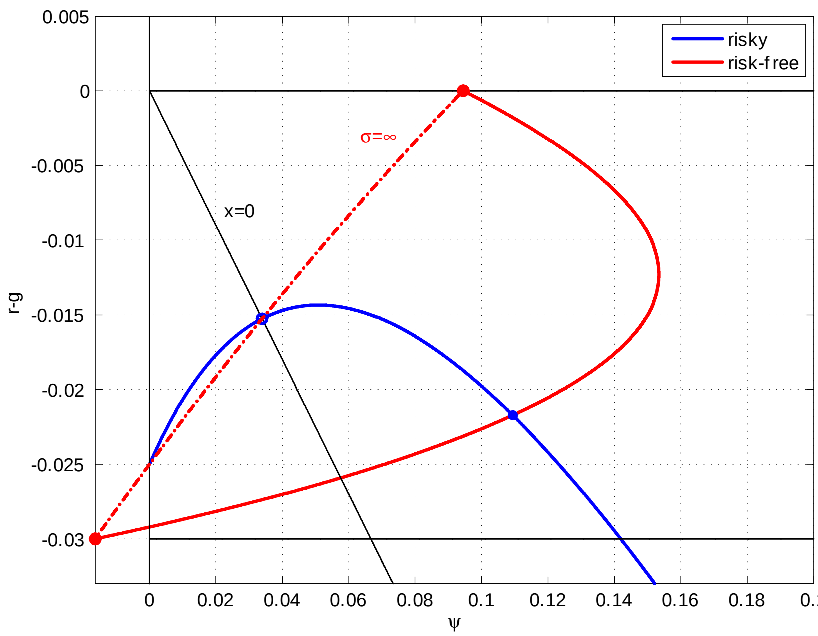

Figure 4 *The equilibrium conditions for $\sigma \in (0, \infty)$ and $\sigma = \infty$.*

As indicated by Figure 4, increasing σ shrinks the risk-free market clearing condition, viewed as a mapping $r - g \mapsto \psi$, towards its large-σ limit. And that limit continues to intersect the risky market clearing condition. In this example, there is no bound on how

[26]The risk-free market clearing condition implied by (25) and (26)-(28) is a quadratic in ψ. It has another branch that is not shown. That branch does not intersect the risky market clearing condition. In Section 5 we provide a more detailed characterization of the risk-free market clearing condition in the $\varepsilon = 1$ special case.

large transfers can be. To explain how it is possible for there to be no bound on the primary deficits a government can run, we focus on $\omega = 0$, so that large $\sigma > 0$ automatically imply large primary deficits.

Consider a sequence of σ that become large, and take a sequence of $\psi \in (0,1)$, $r-g \in (-\delta, 0)$, and $E/W > 0$ that satisfy (23) and (25). Along such a sequence, (23) and (25) imply that $E/W > 0$ must converge to zero. The fact that the portfolio share of government securities has to be bounded above by 1 means that when primary deficits become large as a share of consumer expenditures, consumer wealth must also become large relative to expenditures. Clearly, this is a possibility only if $\varepsilon \neq 1$. Furthermore, the risky market clearing condition (24) implies that, for a converging sequence of deficit equilibria, with $E/W > 0$ converging to zero, $x = \xi\varsigma^2\psi + r - g$ must also converge to zero. If not, then ψ would have to converge to zero, and a strictly positive limit for $x = \xi\varsigma^2\psi + r - g$ together with a zero limit for ψ would imply a strictly positive limit for $r-g$. This would violate the side condition $r - g \in (-\delta, 0)$ that must hold when the government runs a primary deficit. In sum, if there is a converging sequence of deficit equilibria, then it must have the property that both E/W and x converge to zero.

Suppose that it is indeed possible to construct equilibria for all large σ, and suppose that ψ_∞ and $(r-g)_\infty$ are large-σ limits of equilibrium values for ψ and $r-g$. Then the argument just given says that $(r-g)_\infty = -\xi\varsigma^2\psi_\infty$, and that ψ_∞ must solve the quadratic

$$\rho + \delta = \left(1 - \frac{1}{\varepsilon}\right)\left((1-\alpha)\mu + \delta - \omega - \left(1 - (1-\psi_\infty)^2\right) \times \frac{1}{2}\xi\varsigma^2\right).$$

This equation follows from imposing $E/W = 0$ and $x = 0$ in (26)-(28). The factor multiplying $1 - 1/\varepsilon$ on the right-hand side is simply the limiting value of the risk-adjusted consumption growth rate $g_y - \frac{1}{2}\xi\varsigma^2\psi^2$. Since $\rho + \delta > 0$, this risk-adjusted growth rate will have to be negative if $\varepsilon \in (0,1)$, and positive if $\varepsilon \in (1, \infty)$. In any case, the only solution to the quadratic for ψ_∞ that could possibly be in $(0,1)$ is

$$\psi_\infty = 1 - \sqrt{1 - \frac{1}{\xi\varsigma^2/2}\left((1-\alpha)\mu + \delta - \omega - \frac{\rho+\delta}{1-1/\varepsilon}\right)}. \tag{32}$$

This gives ψ_∞ as a strictly increasing function of μ, as long as the right-hand side of (32) is real. The constraint $\xi\varsigma^2\psi_\infty = -(r-g)_\infty < \delta$ implies that $\psi_\infty \in (0, \min\{1, 1/(\xi\varsigma^2/\delta)\})$. By varying ψ_∞ throughout this interval one can trace out the non-empty interval of μ for which there are $\psi_\infty \in (0,1)$ and $(r-g)_\infty \in (-\delta, 0)$ that can be interpreted as large-σ limits of equilibria.

Proposition 7 *Fix some wealth tax $\omega \geq 0$. The economy has an equilibrium for all large enough σ and almost all τ if and only if*

$$0 < \frac{(1-\alpha)\mu + \delta - \omega}{\rho + \delta} - \frac{1}{1 - 1/\varepsilon} < \frac{\delta}{\rho + \delta} \times \begin{cases} \frac{1}{2}\frac{\xi\varsigma^2}{\delta} & \text{if } \frac{\xi\varsigma^2}{\delta} < 1, \\ 1 - \frac{1}{2}\frac{1}{\xi\varsigma^2/\delta} & \text{if } \frac{\xi\varsigma^2}{\delta} > 1. \end{cases} \tag{33}$$

As σ grows without bound, the equilibrium values of ψ and $r - g$ converge to the $\psi_\infty \in (0,1)$ defined in (32) and $(r-g)_\infty = -\xi\varsigma^2\psi_\infty \in (-\delta, 0)$. And E/W and x converge to zero at the same rate, giving rise to zero utility for everyone if $\varepsilon \in (0,1)$, and unbounded utility if $\varepsilon \in (1,\infty)$. The growth rate of aggregate consumption goes to its technological upper bound $g = (1-\alpha)\mu$.

We have only sketched the necessary part of the existence claim in this proposition. The sufficiency part is in the appendix.[27]

The fact that growth goes to its technological upper bound is immediate from the fact that x goes to zero when transfers become large. This also means that the level of aggregate consumption, $C = (xK)^{1-\alpha}L^{1-\alpha}/(1+\gamma)$, goes to zero.[28] If $\varepsilon \in (0,1)$ then (18) implies that U/C goes to zero as E/W goes to zero, and so utility must go to zero. For consumers already alive and consumers who will be born in the near future, very large transfers are the worst possible policy a government can follow. But if $\varepsilon \in (1,\infty)$, then (18) implies that U/C goes to infinity as E/W goes to zero. Furthermore, the risky market clearing condition (24) implies that E/W and x converge to zero at the same rate. Since $1 - \alpha - 1/(1-1/\varepsilon) < 0$ this implies that U/C goes to infinity fast enough to overcome the fact that C goes to zero. In other words, the government can increase U without bound by setting σ large enough. Because $\psi_\infty \in (0,1)$, consumption remains risky, unlike what happens when a government backs its securities with large consumption taxes. Nevertheless, when $\varepsilon \in (1,\infty)$, the risk-adjusted individual consumption growth rate $g_y - \frac{1}{2}\xi\varsigma^2\psi^2$ increases by just enough to make utility explode. Given that wealth taxes are non-negative, (27) together with $\psi_\infty \in (0,1)$ ensures $g_y < g + \delta$ in the limit. So U_y/U has a limit in $(0,\infty)$, and therefore U_y inherits the large-σ limits of U. Everyone will benefit from large transfers.

A comparison of the $\omega = 0$ version of the complete markets bounds (21) and the no-bubble bounds (29) with the $\omega = 0$ version of (33) shows that the region where large transfers without large taxes are possible is precisely the region in between the $\omega = 0$ ver-

[27]The condition (33) is not sufficient only in the knife-edge case $\psi_\infty = (1-\varepsilon)(1-\alpha)\alpha(1+\gamma)/(1+\tau)$.

[28]From (24), aggregate wealth $W = (1-\alpha)K^{1-\alpha}L^\alpha/(\psi x^\alpha)$ goes to infinity. Also, $E/W \downarrow 0$ and $r - g \to (r-g)_\infty \in (-\delta, 0)$ implies that the portfolio share of claims to labor income converges to zero. When transfers are large, most of consumer wealth is invested in risky physical capital and risk-free government securities. A newborn consumer who somehow fails to receive baby bonds would be in dire shape.

sion of (21) and (29). In other words, unbounded permanent primary deficits are possible when an economy with $\omega = 0$ has either a complete markets equilibrium or a no-bubble incomplete markets equilibrium for every $\sigma \geq 0$, but not both.

4.4.2 Pareto Improving Transfers

If $\varepsilon \in (1, \infty)$ and $\omega = 0$, then utility is unbounded if the government can make large transfers to newborn consumers without raising taxes. And the growth rate of the economy converges to its technological upper bound. Under these circumstances, the incomplete markets economy with $\omega = 0$ also has balanced budget equilibrium, without bubble securities, and possibly with bubble securities as well. Clearly, these balanced budget policies are Pareto dominated by permanent deficit policies with large enough transfers to newborn consumers. More generally, for every economy that violates the complete markets bound (21) at $\omega = 0$, it is possible to find $\omega > 0$ so that (33) is satisfied, which then implies unbounded utilities if government transfers to newborn consumers are large. In contrast to the surplus policies described following Proposition 4, large consumption taxes are not needed.

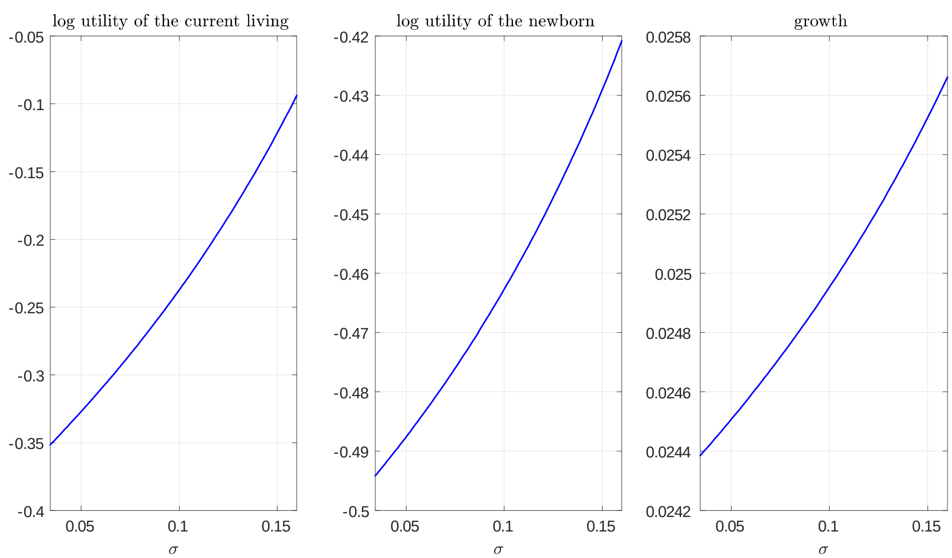

Figure 5 *Pareto improvements from increases in σ*

Proposition 7 only guarantees that large transfers imply large Pareto improvements when $\varepsilon \in (1, \infty)$ and μ satisfies (33). But one can construct robust examples of Pareto improvements that arise from small increases in σ. Figure 5 provides an illustration with $\omega = 0$ and $\varepsilon \in (1, \infty)$ and μ that satisfy (33).[29]

[29]The parameters are $\rho = 0.005$, $\delta = 0.03$, $\varepsilon = 2$, $\xi = 7.5$, $\alpha = 0.6$, $\mu = 0.12$, and $\varsigma = 0.25$.

As in the large-σ limit displayed in Figure 4, the equilibria shown in Figure 5 are the only balanced growth equilibria. The lowest σ shown in Figure 5 is a balanced budget policy. In this example, therefore, Pareto improvements occur as soon as the government begins to use primary deficits to fund transfers to newborn consumers, even though the economy is only growing at a rate that is much below its technological upper bound.[30] It is worth noting that the effects of changes in σ on growth are quite small in our economy.

5 The Special Case $\varepsilon = 1$

Proposition 7 implies that there must be bounds on how large permanent primary deficits can be when $\varepsilon = 1$, no matter how productive or unproductive the economy may be. Here we determine these bounds and describe the effects of taxes and transfers on growth and welfare.

5.1 The Equilibrium Conditions

The equilibrium conditions (23)-(28) depend on ε only via the equilibrium condition (28) for E/W. The assumption $\varepsilon = 1$ implies the familiar and very convenient simplification $E/W = \rho + \delta$. In turn, this means that the surplus ratio $\mathcal{S}_t$ defined in (3) is a parameter, given by

$$\mathcal{S} = \frac{\rho + \delta + \omega}{\rho + \delta} - \frac{(1+\gamma)(1+\sigma)}{1+\tau}.$$

We now allow for wealth subsidies and require ω to satisfy only that $\rho+\delta+\omega > 0$. Suppose the fiscal targets imply $\mathcal{S} \neq 0$, so that (23) gives $D/(PE) = \mathcal{S}/(r-g) \geq 0$. Taking into account that $x = \xi\varsigma^2\psi + r - g$, the market clearing conditions for risky and risk-free assets (24)-(25) then reduce to

$$\frac{\psi}{\rho+\delta} = \frac{1-\alpha}{\xi\varsigma^2\psi + r - g}\frac{1+\gamma}{1+\tau}, \tag{34}$$

$$\frac{1-\psi}{\rho+\delta} = \frac{\alpha}{\delta + r - g}\frac{1+\gamma}{1+\tau} + \frac{\mathcal{S}}{r-g}. \tag{35}$$

The side conditions are $\psi \in (0,1)$ together with $r - g > 0$ if $\mathcal{S} > 0$ and $r - g \in (-\delta, 0)$ if $\mathcal{S} < 0$. If $\mathcal{S} = 0$ and $r - g = 0$, then the term $\mathcal{S}/(r-g)$ on the right-hand side of (35) must be replaced by $D/(PE) \geq 0$. These are now two equilibrium conditions to be solved for

[30] Recall from Proposition 3 that increasing σ when the government is running a surplus in a complete markets economy is good for growth, as it is in Figure 5. In the complete markets economy, there will be winners and losers. Here it is a Pareto improvement.

ψ and $r - g$. As before, x, g, and g_y follow from (26)-(27).

Adding up (34)-(35) and using $\xi\varsigma^2\psi > 0$, $\delta > 0$ and $\sigma \geq 0$ shows that $r-g \in (0, \rho+\delta+\omega)$ if $\mathcal{S} > 0$. So we can infer that $r - g \in (-\delta, \rho + \delta + \omega)$ in any equilibrium.

5.1.1 The Role of the Wealth Tax

The equilibrium values of ψ, $r - g$, x, and g only depend on ω and σ via $\mathcal{S}$, unlike what happens in the $\varepsilon \neq 1$ economy. And the definition of $\mathcal{S}$ shows that there is an unbounded range of wealth taxes $\omega > -(\rho+\delta)$ and transfers $\sigma \geq 0$ of baby bonds that lead to the same surplus ratio $\mathcal{S}$. This means that, in particular, increases in the wealth tax cannot hurt aggregate growth when they are accompanied by budget-neutral increases in transfers to newborn consumers.

But (27) shows that the individual consumption growth rate g_y does depend separately, and negatively, on ω. And aggregate feasibility (13) says that $W_y/W = (g+\delta-g_y)/\delta$. Given rates of return and aggregate quantities, alternative combinations of ω and σ can be used to target the level and growth rate of individual consumption trajectories.

5.1.2 An Easy Corner of the Parameter Space

Suppose consumers are infinitely lived, that labor is not needed to produce consumption, and that there is no wealth tax. That is, δ, α and ω are all zero, and σ as well because there are no newborn consumers to receive transfers. At the cost of not being able to study the role of labor, perpetual youth, and wealth taxes, this produces a very simple equilibrium condition that isolates the role of idiosyncratic risk and incomplete markets.

Eliminating $r - g$ from (34)-(35) then yields

$$\frac{1+\gamma}{1+\tau} = 1 - \left(1 - \frac{\xi\varsigma^2}{\rho} \times \psi^2\right)(1-\psi), \qquad \frac{r-g}{\rho} = 1 - \frac{\xi\varsigma^2}{\rho} \times \psi^2.$$

Given a solution for ψ and $r - g$, $x = \xi\varsigma^2\psi + r - g$ implies $x = \rho + \xi\varsigma^2\psi(1-\psi) > \rho$ and hence the economy grows at the rate $g = \mu - \rho - \xi\varsigma^2\psi(1-\psi) < \mu - \rho$. The cubic on the right-hand side of the equation for ψ is equal to 0 at $\psi = 0$, equal to 1 at $\psi = 1$. It is monotone in between if $\xi\varsigma^2/\rho < 1$ and hump-shaped if $\xi\varsigma^2/\rho > 1$.

This immediately implies a unique equilibrium if $\tau > \gamma$, and this equilibrium satisfies $\psi \in (0,1)$ and $r - g > 0$. The equilibrium must be on the increasing part of $1 - (1 - (\xi\varsigma^2/\rho)\psi^2)(1-\psi)$, and so an increase in $\tau > \gamma$ implies a reduction in ψ. Larger government surpluses reduce the portfolio share of capital. As long as $\psi \in (1/2, 1)$, this lowers the growth rate of the economy. But $\psi \in (0, 1/2)$ if $\xi\varsigma^2/\rho$ is large, and then an increase in the consumption tax increases the growth rate of the economy. In contrast to the results obtained in Proposition 3 for a complete markets economy with perpetual youth, the sign of the effect on growth of changing consumption taxes while the government runs a surplus depends on parameters.

At $\tau = \gamma$, the cubic for ψ has two positive solutions, $\psi = 1$ and $\psi = 1/\sqrt{\xi\varsigma^2/\rho}$, and hence $r - g = 1 - \xi\varsigma^2/\rho$ or $r - g = 0$. If $\psi = 1$ then be value of government securities is zero, and we have a no-bubble equilibrium. Precisely when $r - g < 0$ in this no-bubble equilibrium, the economy also has a bubble equilibrium, in which the value of government securities is positive and $r - g = 0$. This happens when $\xi\varsigma^2/\rho > 1$. In that case, $1 - (1 - (\xi\varsigma^2/\rho)\psi^2)(1 - \psi) > 1$ in between the two equilibria, and so there will also be two equilibria if the government runs a primary deficit that is not too large.

5.2 The Three Primary Surplus Scenarios

Figure 6 shows the equilibrium conditions (34)-(35) for a common $(1+\gamma)/(1+\tau) < 1$, and with $\sigma \geq 0$ selected to illustrate each of the three possible primary surplus scenarios.

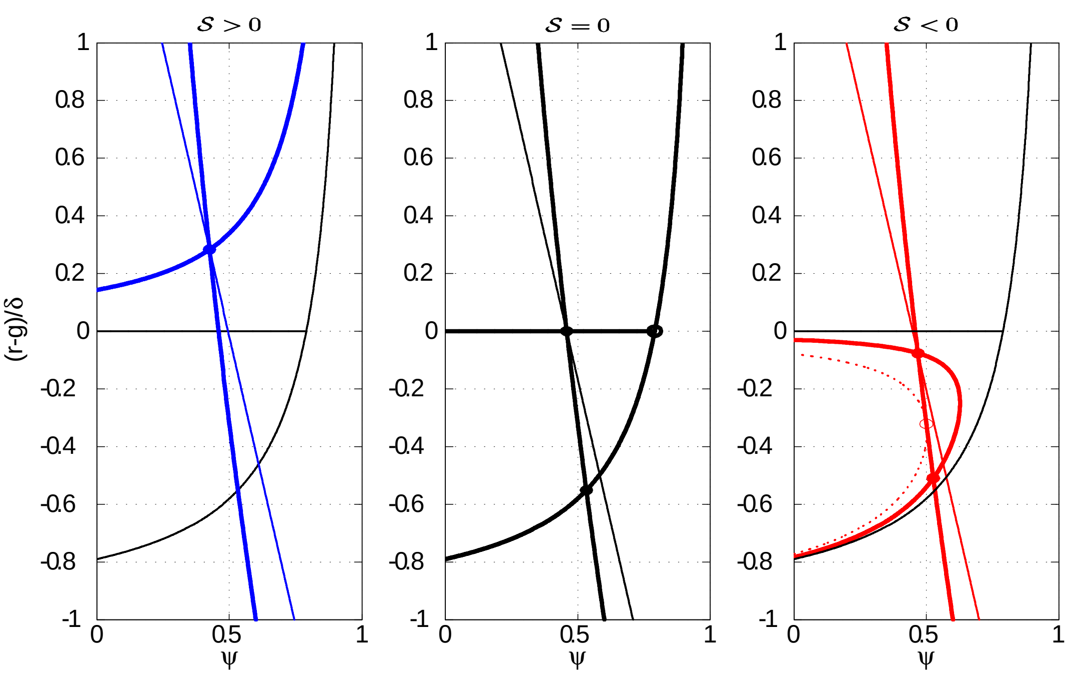

Figure 6 *Equilibria for the three primary surplus scenarios.*

The risky market clearing condition (34) is the thick downward-sloping curve in each of the three panels of Figure 6. This curve is a hyperbola, with a vertical asymptote at $\psi = 0$, and a large-ψ asymptote $-\xi\varsigma^2\psi$. It gives $r - g$ as a convex function of $\psi \in (0, 1)$. The large-ψ asymptote has a strictly negative slope precisely because $\xi\varsigma^2 > 0$. In the absence of idiosyncratic risk (or when markets are complete), the curve (34) would remain positive for all $\psi \in (0,1)$, ruling out the equilibria shown in the $\mathcal{S} < 0$ panel of Figure 6. The risk-free market clearing condition (35) takes on a different shape depending on whether $\mathcal{S} > 0$, $\mathcal{S} = 0$, or $\mathcal{S} < 0$.

Primary Surpluses If $\mathcal{S} > 0$, then the right-hand side of (35) is large for $r - g$ close to zero, strictly decreasing in $r - g > 0$, and converging to zero as $r - g$ becomes large. As a result, (35) defines an increasing function that maps $\psi \in (0,1)$ into $r - g \in (0,\infty)$. This is the thick upward sloping curve in the $\mathcal{S} > 0$ panel of Figure 6. It is easy to see that (34) and (35) must always intersect when $\mathcal{S} > 0$, and that these curves intersect only once. The equilibrium is unique.

Balanced Budgets The upward-sloping curve in the $\mathcal{S} = 0$ panel is obtained by setting $\mathcal{S} = 0$ in (35) and varying $r - g \in (-\delta, \infty)$. The result is an upward sloping and convex hyperbola with asymptotes at $r - g = -\delta$ and $\psi = 1$. Because $D/(PE)$ can be any non-negative number when budgets are balanced, the $\mathcal{S} = 0$ version of the equilibrium condition (35) also includes the positive horizontal axis up to the point where (35) crosses the horizontal axis. In the example of Figure 6, this leads to two equilibria: a bubble equilibrium with $r - g = 0$, and a no-bubble equilibrium with $r - g < 0$. Increasing σ so that $\mathcal{S} \downarrow 0$ causes the $\mathcal{S} > 0$ version of (35) to converge to the $r - g \geq 0$ segment of the $\mathcal{S} = 0$ version of (35). This means that the unique equilibrium for $\mathcal{S} > 0$ converges to the bubble equilibrium for $\mathcal{S} = 0$.

For reference, the $\mathcal{S} = 0$ version of the risk-free market clearing condition is also shown in the background of the $\mathcal{S} > 0$ and $\mathcal{S} < 0$ panels.

Primary Deficits For $\mathcal{S} < 0$, the risk-free market clearing condition (35) is given by the hump-shaped curve that maps $r - g \in (-\delta, 0)$ into $\psi < 1$ in the third panel of Figure 6. Its shape arises because $\mathcal{S} < 0$ implies that the right-hand side of (35) is a convex function of $r - g \in (-\delta, 0)$, with vertical asymptotes at $r - g = -\delta$ and $r - g = 0$. It is the sum of a present value of labor income that is decreasing in $r - g > -\delta$, and a steady state value of government securities that is increasing in $r - g < 0$. Holding fixed $(1+\gamma)/(1+\tau)$, it is easy to see that there is no limit on how far this curve can shift to the left as $-\mathcal{S}$ becomes large, by taking σ to be large. Since ψ must be in $(0,1)$ in any equilibrium, this implies a weak upper bound on $-\mathcal{S}$ given $\sigma \geq 0$, even without reference to (34).[31]

[31] This upper bound can be written implicitly as $\sqrt{\alpha\,(1-\mathcal{S})\,/(1+\sigma)} + \sqrt{-\mathcal{S}} < \sqrt{\delta/(\rho+\delta)}$. Impatient consumers and a high labor share tighten this upper bound.

Taking into account both equilibrium conditions, the upper bound on σ given $(1+\gamma)/(1+\tau)$ can be found by increasing σ to the point where (34) and (35) are tangent. Such a tangency can also be used to characterize the lower bound on $\tau \in (-1, \infty)$ given $(1+\gamma)(1+\sigma)$. Reductions in τ shift (34) to the right and (35) to the left, directly via an increase in the value of claims to labor income, and indirectly via an in increase in $-\mathcal{S}$.[32]

As Figure 6 suggests, lowering σ for a given $(1+\gamma)/(1+\tau) < 1$ so that $\mathcal{S} \uparrow 0$ causes the $\mathcal{S} < 0$ version of (35) to converge to the $r - g \leq 0$ segment of the $\mathcal{S} = 0$ version of (35).

The Effects on Growth Given ψ and $r-g$ that solve (34)-(35), the equilibrium growth rate g follows from $x = \xi\varsigma^2\psi + r - g$ and $g = (1-\alpha)(\mu - x)$, and then the risk-free rate is $r = g + r - g$. The thin downward sloping lines in Figure 6 represent lines with a constant value of $x = \xi\varsigma^2\psi + r - g$. All outcomes along these lines result in the same level of consumption and the same growth rate. It is important to note that, where they cross, the curve $\psi \mapsto r - g$ implied by the risky market clearing condition (34) must always be steeper than the line $r - g = x - \xi\varsigma^2\psi$. It has an additional term that scales with $1/\psi$.

As long as $\mathcal{S} > 0$, an increase in σ shifts (35) to the right along the fixed downward sloping curve (34). As Figure 6 shows, this lowers x and therefore increases the growth rate of the economy. This is already familiar from the complete markets economy. Holding fixed $(1+\gamma)/(1+\tau)$, an economy with a surplus grows more slowly than an economy with a balanced budget, for both of the two balanced budget equilibria. If the equilibrium changes continuously with further increases in σ, then growth will increase further as long as targeted deficits are still consistent with equilibrium.

Similarly, a budget neutral increase in transfers moves (34) to the left, and (35) to the right. As in the complete markets economy, this lowers x and therefore increases growth when $\mathcal{S} \geq 0$. In other words, when the government does not run a deficit, an increase in the consumption tax that is used to fund transfers is good for growth.

5.3 Balanced Budgets and Bubbles

Contrary to the illustration given in the $\mathcal{S} = 0$ panel of Figure 6, the downward sloping curve (34) may well intersect the curve (35) to the right of where both of these curves intersect the horizontal axis. In that case, there is a unique equilibrium, and it has $r - g > 0$. There is no bubble equilibrium. We now show how these two possible scenarios depend on parameters of the economy and fiscal targets.

[32]Suppose $\rho = 0.005$, $\delta = 0.03$, $\mu = 0.1$, $\alpha = 5/9$, and $\xi\varsigma^2 = 7.5(0.3)^2$. Starting from $\gamma = 1/20$, $\tau = 1/5$, and $\sigma = 1/7$, varying only σ or only τ gives a maximal deficit ratio $-\mathcal{S}$ equal to about 0.025.

To construct a possible bubble equilibrium, set $r - g = 0$, and use (34) to infer that

$$\psi = \sqrt{\frac{1-\alpha}{\xi\varsigma^2/\delta}\frac{\rho+\delta}{\delta}\frac{1+\gamma}{1+\tau}}.$$

Plugging this into (35) gives

$$\frac{D}{PE} = \frac{1}{\rho+\delta}\left(1 - \left(\alpha \times \frac{\rho+\delta}{\delta}\frac{1+\gamma}{1+\tau} + \sqrt{\frac{1-\alpha}{\xi\varsigma^2/\delta}\frac{\rho+\delta}{\delta}\frac{1+\gamma}{1+\tau}}\right)\right). \tag{36}$$

In Figure 6, the value $D/(PE)$ of the bubble is measured by the distance on the horizontal axis between the points where (34) and (35) cross the horizontal axis (marked by a solid dot and a circle, respectively). The properties of the balanced budget economy are summarized in the following proposition, which is proven in the appendix.

Proposition 8 *When fiscal targets imply balanced budgets, the economy has a unique steady state equilibrium without a bubble. This no-bubble equilibrium has $r - g < 0$ if and only if the right-hand side of (36) is positive. The economy then also has a unique steady state equilibrium with a strictly positive bubble. When $\omega = 0$ and $\sigma = 0$, the requirement that (36) is positive is equivalent to*

$$\alpha < \frac{\delta}{\rho+\delta}\left(1 - \frac{\delta/2}{\xi\varsigma^2} - \sqrt{\left(\frac{\delta/2}{\xi\varsigma^2}\right)^2 + \frac{\rho}{\xi\varsigma^2}}\right). \tag{37}$$

More generally, an economy with $\mathcal{S} = 0$ has a pure bubble equilibrium if and only if $(1+\gamma)/(1+\tau)$ satisfies

$$\alpha \times \frac{1+\gamma}{1+\tau} < \frac{\delta}{\rho+\delta}\left(-\frac{1}{2}\sqrt{\frac{\delta}{\xi\varsigma^2}\frac{1-\alpha}{\alpha}} + \sqrt{\left(\frac{1}{2}\sqrt{\frac{\delta}{\xi\varsigma^2}\frac{1-\alpha}{\alpha}}\right)^2 + 1}\right)^2. \tag{38}$$

The economy grows faster in the no-bubble equilibrium than in the bubble equilibrium.

In the special case given by $\omega = 0$ and $\sigma = 0$, the upper bound (37) on the labor share parameter α is decreasing in $\xi\varsigma^2$ and positive if and only if $\xi\varsigma^2 > \rho+\delta$. As in the economy infinitely lived consumers and no fixed factor, there must be enough idiosyncratic risk. But here the labor share also plays an important role. No amount of idiosyncratic risk makes a bubble possible if $\alpha > \delta/(\rho+\delta)$.

On the other hand, when $\sigma \geq 0$ and $\tau > 0$ can be large subject to $\mathcal{S} = 0$, then (38) says that for any $\xi\varsigma^2 > 0$, there will be a bubble equilibrium as long as transfers and consumption taxes are high enough. This is already clear from the fact that the expression for $D/(PE)$ given in (36) is positive when τ is large enough. Lowering $(1+\gamma)/(1+\tau)$ shifts the risky market clearing condition (34) to the left and the no-bubble version of the risk-free market clearing condition (35) to the right. The result is a lower $r-g$, and once $r-g$ becomes negative, this makes a bubble possible.[33]

Large Taxes and Transfers imply Maximal Growth As τ and σ become large subject to $\mathcal{S}=0$, (34) converges to $\psi = \max\{0, -(r-g)/(\xi\varsigma^2)\}$ and the $r-g \neq 0$ branch of (35) to the line segments $\{(\psi, r-g) : \psi \in [0,1], r-g = -\delta\}$ and $\{(\psi, r-g) : \psi = 1, r-g > -\delta\}$. It follows that the bubble equilibrium converges to $(\psi, r-g) = (0,0)$. This maximizes the size of the bubble, as predicted by (36). This also means that x goes to zero. In other words, the growth rate of the economy converges to its technological upper bound, and consumer exposure to idiosyncratic risk disappears. The no-bubble equilibrium converges to $\psi = \min\{1, 1/(\xi\varsigma^2/\delta)\}$ and $r-g = -\min\{\xi\varsigma^2, \delta\}$. This again means maximal growth, but now consumers remain exposed to idiosyncratic risk.

5.4 The Maximal Deficit Ratio

Figure 6 already shows that there can be no permanent primary deficits if the $\mathcal{S}=0$ economy does not have a bubble equilibrium. Proposition 8 shows that there will be bubble equilibria in the $\mathcal{S}=0$ economy when consumption taxes and transfers to newborn consumers are large enough. We have already argued that, when $\mathcal{S}<0$ is actually possible, there will be an upper bound on σ given τ, and a lower bound on τ given σ. Here we find an explicit expression for the largest possible primary deficit ratio $-\mathcal{S}$ that can result when consumption taxes and transfers are varied jointly.

Proposition 9 *For any surplus ratio that satisfies*

$$0 < \mathcal{S} + \frac{\delta}{\rho+\delta} \times \begin{cases} \frac{1}{2} \times \frac{\xi\varsigma^2/2}{\delta} & \text{if} \quad \frac{1}{2}\xi\varsigma^2 < \delta \\ 1 - \frac{1}{2} \times \frac{\delta}{\xi\varsigma^2/2} & \text{if} \quad \frac{1}{2}\xi\varsigma^2 > \delta \end{cases}$$

there are fiscal targets for which the economy has an equilibrium. As $-\mathcal{S}$ approaches this upper bound, consumption taxes and transfers to newborn consumers have to become large, and the growth rate approaches its maximal feasible rate.

[33]This is reminiscent of what happens in an exchange economy with two-period lived consumers and overlapping generations. A transfer to the young financed by a consumption tax can make the interest rate at which the young are willing to save negative.

When there is a large amount of idiosyncratic risk, this says that the upper bound on $-\mathcal{S}$ is approximately $\delta/(\rho+\delta)$, implying primary deficits almost as large as aggregate consumption expenditures when ρ is very close to zero. But even at $\xi\varsigma^2/2 = \delta$, this bound says that the upper bound on $-\mathcal{S}$ is as large as half of aggregate consumption expenditures when $\rho = 0$.

To prove this proposition, use (34) to eliminate $(1+\gamma)/(1+\tau)$ from the first term in (35). The result is an equation that maps ψ and $r-g$ into $\mathcal{S}$. Varying $\psi \in (0,1)$ and $r-g \in (-\delta, 0)$ subject to $\xi\varsigma^2\psi + r - g > 0$ then gives the feasible range for $\mathcal{S}$. Subject to these constraints, the supremum of $-\mathcal{S}$ is approached by letting $x = \xi\varsigma^2\psi + r - g \downarrow 0$ and $r - g \downarrow -\min\{\delta, \xi\varsigma^2/2\}$. The fact that $x \downarrow 0$ means that aggregate growth is maximal. The fact that large consumption taxes will be required in such a limit is immediate from (34). A detailed version of this backsolving argument is in the appendix.

5.5 Primary Deficits in the UBI Economy

The UBI version of (34) is the same as in the baby bonds economy. In the UBI version of (35), σ must be replaced by θ, and α by $\alpha+\theta$. Define

$$\mathcal{S}_* = 1 - \frac{\rho+\delta}{\rho+\delta+\omega}\frac{(1+\gamma)(1+\theta)}{1+\tau}.$$

Note that $\mathcal{S}_* = \mathcal{S}$ when $\omega = 0$ and $\theta = \sigma$. The UBI version of (34)-(35) is then

$$\begin{aligned}
\frac{\psi}{\rho+\delta+\omega} &= \frac{1-\frac{\alpha+\theta}{1+\theta}}{\xi\varsigma^2\psi + r - g} \times (1-\mathcal{S}_*), \\
\frac{1-\psi}{\rho+\delta+\omega} &= \frac{\frac{\alpha+\theta}{1+\theta}}{\delta + r - g} \times (1-\mathcal{S}_*) + \frac{\mathcal{S}_*}{r-g}.
\end{aligned}$$

This is of the exact same form as (34)-(35) with $\omega = 0$ and $\sigma = 0$. The upper bound (37) therefore applies to $(\alpha+\theta)/(1+\theta) \in (\alpha, 1)$ rather than to α itself. This immediately implies that there can only be a bubble equilibrium if θ is not too large. Certainly, $(\alpha+\theta)/(1+\theta) < \delta/(\rho+\delta+\omega)$ is necessary, and this is sufficient only if $\xi\varsigma^2/\delta$ is large. This is in sharp contrast to the fact that a large baby bonds parameter σ can ensure the existence of a bubble equilibrium. Furthermore, taking the UBI parameter θ to be positive lowers the maximal size of the steady state primary deficit that is consistent with equilibrium.

Proposition 10 *If transfers are in the form of a universal basic income, then, holding fixed ω, the largest possible steady state primary deficit is attained by setting the universal basic income to zero.*

The resulting upper bound on $-\mathcal{S}_*$ is the same as the bound already described for the baby bonds economy with $\sigma = 0$. The risk-free income generated by a UBI competes with the government's ability to sell risk-free securities, while transfers to newborn consumers, to some extent, generate their own demand for risk-free securities. An increase in the UBI tightens the upper bound on primary deficits faced by the government. If primary deficits are close to their upper bound, a non-trivial increase in the UBI must therefore be accompanied by some combination of higher consumption taxes and lower government purchases that exceeds the extent to which the higher UBI would raise the deficit.

"Fiscal Space" The combination of the basic comparative statics of Figure 6 and of Propositions 9 and 10 highlights the fact that how large primary deficits can be depends very much on how these deficits are used. The upper bound on $-\mathcal{S}$ given in Proposition 9 is not some sort of budget constraint. There is no sense in which there is a single notion of "fiscal space" that the government can use for whatever purpose it chooses.

5.6 Growth and Welfare

We have already shown that balanced budget or deficit equilibria with large taxes and transfers can be very good for growth. Generations of consumers who will be born far into the future will certainly benefit. But policies that maximize the growth rate of the economy will be far from desirable for consumers already alive and nearby generations of newborn consumers. Here we describe equilibrium allocations that are not Pareto dominated by other equilibrium allocations under alternative assumptions about the policy instruments available to the government.

5.6.1 Stationary Utilities

The utilities for this economy can be obtained by taking the $\varepsilon \to 1$ limit in (9) and (11). Since $C = (xK)^{1-\alpha}L^\alpha/(1+\gamma)$ this implies that

$$U = \frac{(xK)^{1-\alpha}L^\alpha}{1+\gamma} \exp\left(\frac{g_y - \frac{1}{2}\xi\varsigma^2\psi^2}{\rho+\delta}\right), \quad U_y = \frac{W_y}{W} \times U, \tag{39}$$

where g_y and W_y/W are given by

$$g_y = g + \delta + x - \xi\varsigma^2(1-\psi)\psi - (\rho+\delta+\omega), \quad \frac{W_y}{W} = \frac{g+\delta-g_y}{\delta}, \tag{40}$$

and $g = (1-\alpha)(\mu - x)$. Unlike what we have seen in sufficiently productive economies with $\varepsilon > 1$, here there is no way to keep U and U_y away from zero when g approaches its technological upper bound. It will be useful to note that the risk adjustment $-\frac{1}{2}\xi\varsigma^2\psi^2$ in (39) and the contribution $\psi(\mu + \mu_q - r) = \xi\varsigma^2\psi^2$ to individual consumption growth g_y in (40) are small when $\psi \in (0,1)$ is small. But the negative effect of ψ on g_y via the risk-free rate $r = (1-\alpha)\mu + \alpha x - \xi\varsigma^2\psi$ is of first order in ψ.

5.6.2 Fixed Taxes and Varying σ Only

Consider the effects of varying σ while holding $\omega \geq 0$ and $(1+\gamma)/(1+\tau) \leq 1$ fixed. We know from Proposition 7 and Figure 5 that increases in σ can lead to Pareto improvements when $\varepsilon \in (1, \infty)$ and the economy is sufficiently productive. Here we show that this cannot happen when $\varepsilon = 1$.

Figure 7 shows the problem: the utility of consumers already alive and the growth rate of the economy always move in opposite directions. In other words, even without considering newborn consumers, varying transfers cannot be used to create Pareto improvements. Any attempt to improve the steady state utility of consumers already alive comes at the cost of lower growth, and that will hurt generations far enough into the future.

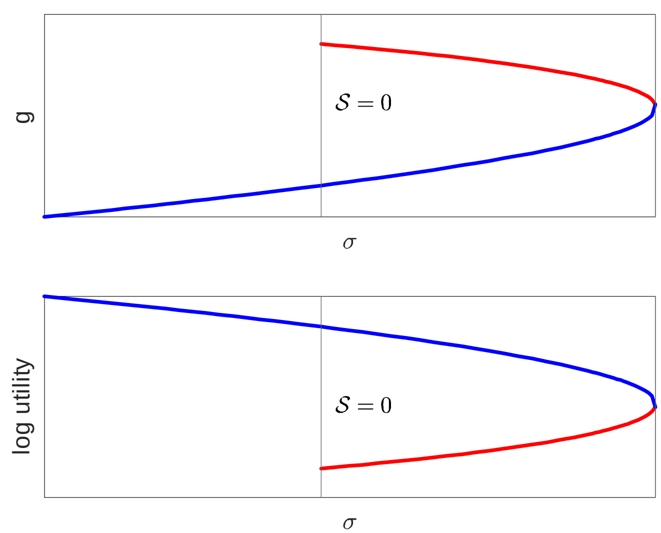

Figure 7 *The welfare consequences of varying σ*

To see why this is true, begin by observing that varying σ implies changes in the surplus ratio $\mathcal{S}$ that lead to shifts in the risk-free market clearing condition (35) along a fixed risky market clearing condition (34). Starting with some $\mathcal{S} > 0$, increasing σ lowers $\mathcal{S}$ and shifts the risk-free market clearing condition (35) to the right, towards the $\mathcal{S} = 0$ version of that condition. As can be seen using Figure 6, this increases ψ and lowers both $r - g$ and x. Since $-(1-\psi)\psi - \frac{1}{2}\psi^2$ is decreasing in $\psi \in (0,1)$ and $g + x = (1-\alpha)\mu + \alpha x$ is increasing in x, this implies a reduction in U. But $g = (1-\alpha)(\mu - x)$ increases with the reduction in x, and so newborn consumers sufficiently far into the future will gain.

Next, suppose (38) holds, so that the economy will have both a no-bubble equilibrium and a bubble equilibrium when $\mathcal{S}$ reaches zero. Consider the equilibria with $r - g < 0$ close to zero that emerge when $\mathcal{S}$ becomes negative as a result of further increases in σ. For these equilibria, as (35) shrinks towards the vertical axis in Figure 6, the increases in σ further lower $r - g$, increase ψ, and lower x. The result is further reductions in U and increases in g. This continues until $-\mathcal{S}$ reaches its upper bound. At $\mathcal{S} = 0$, the no-bubble equilibrium also has a lower $r - g$, a larger ψ, and a lower x than the bubble equilibrium. This implies a worse outcome for U and a better outcome for g in the no-bubble equilibrium than in the bubble equilibrium. It is easy to verify that the same conflict of interest arises for the $\mathcal{S} < 0$ equilibria with $r - g < 0$ close to $-\delta$.

This proves the following proposition and explains the example shown in Figure 7.

Proposition 11 *Holding fixed consumption and wealth taxes, increases in transfers to newborn consumers lead to lower and more risky individual consumption growth, a lower level of aggregate consumption, and faster aggregate consumption growth if $r - g$ is positive or negative and relatively close to zero. The opposite happens when $r - g$ is negative and relatively close to $-\delta$. This leads to a conflict of interest between consumers already alive and consumers who will be born sufficiently far into the future.*

5.6.3 No Wealth Taxes and Varying σ and τ Only

Now take $\omega = 0$ and suppose that the government can vary both σ and τ. We already know from Proposition 4 that by choosing very large consumption taxes and transfers to newborn consumers, the government can approximate any of the stationary allocations that are Pareto efficient. Here we highlight the fact that all current and future newborn generations will then be worse off than they would be with more limited government interventions that do not eliminate all idiosyncratic risk.

The absence of a wealth tax implies that g_y is tied down by ψ and x. It is easy to see from (39)-(40) that $\partial U/\partial\psi < 0$ and $\partial U/\partial x > 0$. A higher ψ implies more risk and reduces the risk-adjusted growth rate $g_y - \frac{1}{2}\xi\varsigma^2\psi^2$. And a higher x implies both a higher level of consumption and higher expected rates of return, because slow aggregate growth implies a low rate at which capital depreciates. Since any equilibrium must have $\psi \in (0,1)$ and

$r-g \in (-\delta, \rho+\delta)$, this means that consumers already alive want to be as close as possible to $\psi = 0$ and $x = \rho + \delta$. This is also the complete markets equilibrium preferred by these consumers.

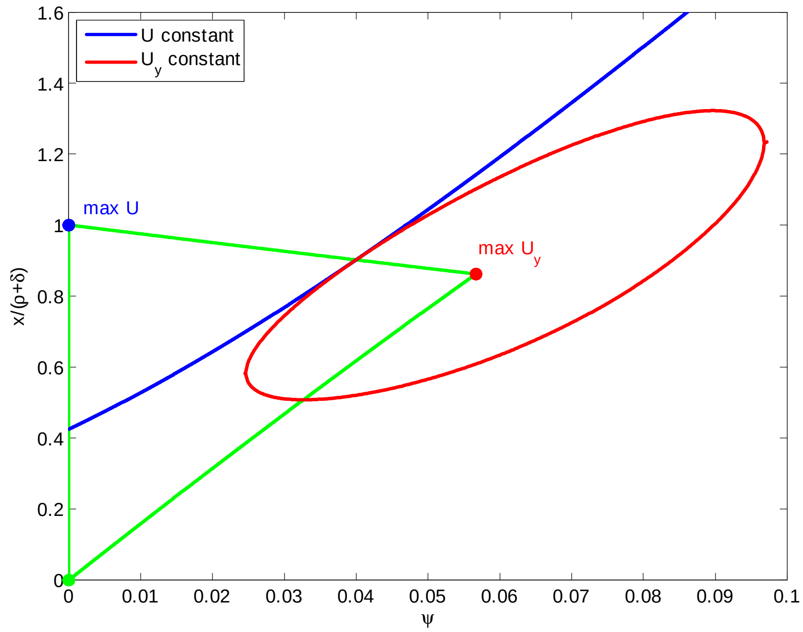

Figure 8 *Indifference curves and contract curves*

But the corner allocation $\psi = 0$ and $x = \rho + \delta$ implies $g_y = g + \delta$, and hence $W_y/W = 0$, the worst possible outcome for newborn consumers. Combining (40) with $g = (1-\alpha)(\mu - x)$ shows that $\partial(W_y/W)/\partial\psi > 0$ if and only if $\psi \in (0, 1/2)$, and that $\partial(W_y/W)/\partial x < 0$. So the effects of $\psi \in (0, 1/2)$ and $x > 0$ on W_y/W are the opposite of those on U. For low ψ and high x, these effects are strong enough to create disagreement between consumers already alive and the current generation of newborn consumers about the desirability of reducing ψ or increasing x. In particular, the utility U_y at $\psi = 0$ is strictly increasing in ψ and hump-shaped in $x \in (0, \rho+\delta)$. One can show that U_y has a unique maximum, and that it satisfies $\psi \in (0, 1)$ and $x \in (0, \rho + \delta)$. Relative to the allocation preferred by consumers already alive, newborn consumers want to take some risk and increase aggregate growth in order to improve their share W_y/W in aggregate consumption. We have already seen in Figure 2 that the most preferred complete markets equilibrium for newborn consumers has a strictly positive wealth tax, resulting in an allocation that is therefore not Pareto efficient. Here we learn that, without a wealth tax, newborn consumers prefer to accept some amount of idiosyncratic risk even though this is not Pareto efficient and even though the government does have the instruments to eliminate it.

Figure 8 shows indifference curves for U and U_y at an allocation where U_y is maximal given a lower bound on U. This allocation is on a curve that connects the allocation that maximizes U to the allocation that maximizes U_y. This is the contract curve for consumers already alive and the current generation of newborn consumers. Also shown is a curve that connects the allocation that maximizes U_y to the origin. This curve is defined by $\partial U_y/\partial\psi = 0$ and x below what is optimal for newborn consumers. Since $g = (1-\alpha)(\mu - x)$ does not depend on ψ, this is the contract curve between newborn consumers and consumers who will be born arbitrarily far into the future. In fact, every point on this contract curve maximizes $\ln(U_y) + gT$ for some $T \in (0,\infty)$. The vertical axis between $x = 0$ and $x = \rho+\delta$ is also a contract curve, for consumers already alive and consumers that will be born very far into the future. These three contract curves enclose a triangle-like area. Inside this area, the indifference curves of U and U_y cross, but always in such a way that increasing U and U_y requires an increase in x. Such an increase lowers growth, and therefore makes consumers who will be born sufficiently far into the future worse off. In other words, the equilibrium allocations in this area are not Pareto dominated by any other equilibrium allocations.

This sketches a proof of the following proposition. The full proof is in the appendix.[34]

Proposition 12 *An equilibrium allocation (ψ,x) is not Pareto dominated by another equilibrium allocation if and only if*

$$\frac{\partial U_y}{\partial\psi} \geq 0 \quad \textit{and} \quad -\frac{\partial U/\partial x}{\partial U/\partial\psi} \geq -\frac{\partial U_y/\partial x}{\partial U_y/\partial\psi}. \tag{41}$$

The two inequalities in (41) can be written as, respectively,

$$(1-\psi)\psi \times \frac{\xi\varsigma^2}{\rho+\delta} + \frac{\psi}{1-\psi} \leq \frac{x}{\rho+\delta} \leq 1 - \frac{\psi}{(1-\alpha)(1-\psi)+\alpha\psi}. \tag{42}$$

In turn, (42) implies that $\psi \in (0,1/2)$, $r-g>0$, and $x<\rho+\delta$ are all necessary for an equilibrium allocation not to be Pareto dominated by another equilibrium allocation.

The necessary condition $\psi \in (0,1/2)$ is equivalent to $\partial(W_y/W)/\partial\psi > 0$. Since $\partial U/\partial\psi < 0$, this is necessary for $\partial U_y/\partial\psi \geq 0$. Unlike in an economy with complete markets, the condition $r-g>0$ is not sufficient for an allocation not to be Pareto dominated by another equilibrium allocation.[35] The proposition shows that the $x<\rho+\delta$ property of

[34] In the example of Figure 8, all outcomes in the region bounded by the three contract curves, minus the vertical axis, are indeed implementable. But for low $\xi\varsigma^2 > 0$ and high $\alpha \in (0,1)$, it is possible for a subset of allocations in this region to violate the constraint that the government does not lend to the public. This shrinks but does not empty the set of efficient allocations that are implementable. All allocations with $x \in (0,\rho+\delta)$ and ψ close enough to zero are implementable.

[35] A sharper implication of (42) is $\frac{r-g}{\rho+\delta} > \frac{\psi}{1-\psi}\left(\frac{\psi}{1-\psi} + \frac{\psi}{(1-\alpha)(1-\psi)+\alpha\psi}\right)$. This lower bound can be inferred

Pareto efficient equilibria also generalizes.

5.6.4 Varying σ, τ, and Wealth Taxes

For fixed x, a reduction in ψ implemented using σ and τ has a common benefit for U and U_y because of the risk-adjustment $-\frac{1}{2}\xi\varsigma^2\psi^2$ to individual consumption growth. But for $\psi \in (0, 1/2)$, a reduction in ψ also has a positive first-order effect on the risk-free rate that dominates the negative second-order effect on the risky component of the expected return on wealth. As a result, a reduction in $\psi \in (0, 1/2)$ increases g_y. Note that $(\rho + \delta) \times \partial \ln(U)/\partial g_y = 1$ and $(\rho + \delta) \times \partial \ln(U_y)/\partial g_y = 1 - (\rho + \delta)/(g + \delta - g_y)$. This means that $\partial U_y/\partial g_y$ is negative whenever $\rho + g_y > g$ and g_y is feasible, as is certainly the case inside the triangle-like region of Figure 8.

Now suppose the government can also increase the wealth tax ω. Because a budget-neutral increase in ω and σ leaves ψ, x, and therefore g unchanged, the government can raise ω to undo the positive effect of a reduction in ψ on g_y. Only the common benefit of a reduced risk-adjustment $-\frac{1}{2}\xi\varsigma^2\psi^2$ then remains, which makes the combined change in ψ and ω a Pareto improvement for consumers already alive and newborn consumers.

This argument applies as long as ψ is positive. If there are bounds on how large taxes and transfers can be, then it may not be possible to take ψ arbitrarily close to zero. But without such bounds, Proposition 4 applies, and the government can use large transfers and consumption taxes to approximate all complete markets equilibria with $\omega \geq 0$, ranging from the Pareto efficient equilibria defined by $\omega = 0$, all the way to the complete markets equilibrium preferred by the current generation of newborn consumers. In such a setting, both $r - g < 0$ and any evidence of significant idiosyncratic risk are immediate indications of an inefficient configuration of government policy.

More generally, all stationary allocations that are not Pareto dominated by other stationary allocations can be approximated when both wealth taxes and wealth subsidies are feasible. These are the allocations in the quadrilateral shown in Figure 2, which here becomes a parallelogram. To see this, take some $\omega \in (-(\rho + \delta), \rho + \delta)$ and a surplus ratio $\mathcal{S} \in (0, (\rho + \delta + \omega)/(\rho + \delta))$. The equilibrium is unique, with $r - g > 0$. Holding fixed ω and $\mathcal{S}$, one can let both τ and σ become large. This makes the right-hand side of the risky market clearing condition (34) go to zero at any $\psi \in (0, 1)$ and $r - g > 0$. The first term on the right-hand side of the risk-free market clearing condition (35) also goes to zero. The limit of the resulting sequence of unique equilibria is therefore $\psi = 0$ and $r - g = (\rho + \delta)\mathcal{S}$, which gives $x = (\rho + \delta)\mathcal{S}$ and some growth rate $g = (1 - \alpha)(\mu - x)$. Con-

from the portfolio share of risk-free assets and the labor share in the consumption sector only, without the need to measure $\xi\varsigma^2$.

sumption becomes risk-free in this limit, and the growth rate of individual consumption $g_y = x + g + \delta - (\rho + \delta + \omega)$ satisfies $g_y < g + \delta$ by construction.

By using large σ and τ, and varying ω and $\mathcal{S}$, one can now trace out the parallelogram in Figure 2. The best stationary allocation for consumers already alive is approximated by $\omega = 0$ and $\mathcal{S} \uparrow 1$, and the best stationary allocation for the current generation of newborn consumers requires $\omega = \rho + \delta$ and $\mathcal{S} \uparrow 1$. Letting $\omega = 0$ and varying $\mathcal{S} \in (0, 1)$ allows a government to approximate all stationary allocations that are also Pareto efficient.

6 Finitely Lived Consumers

The assumption that consumers die randomly at some rate $\delta > 0$ plays an absolutely critical role in generating the possibility of unbounded utilities. This perpetual youth assumption is clearly a bad assumption for individual consumers. But the consumers in this economy can also be viewed as dynastically linked individuals who care about their descendants (Weil [1989]). The rate δ can then be interpreted as the rate at which altruistic links break down. If $\varepsilon \in (1, \infty)$ and the conditions of Proposition 7 apply, then potential dynastic utilities are unbounded.

Consider the other extreme: consumers who live finite lives and who do not care about their descendants. Specifically, suppose consumers die randomly at the rate δ, and for certain when they reach the age $T > 0$. The flow of new births is $\delta/(1 - e^{-\delta T})$, which implies a unit measure of consumers in the steady state. There is no bequest motive, and so consumers will choose to spend all their wealth by the time they reach age T.

In this setting we show that it is possible to construct fiscal policies so that the equilibrium utilities in the finite-T economy approximate their $T = \infty$ counterparts.

6.1 Decision Rules and Aggregation

For consumers faced with constant rates of return, the Epstein-Zin preferences we have used all along again give rise to the portfolio choice $\psi = (\mu + \mu_q - r)/(\xi\varsigma^2)$. But the optimal consumption-wealth ratio does depend on age.[36] It is of the form

$$\phi_a = \frac{\phi_\infty}{1 - e^{-\phi_\infty (T-a)}}, \quad a \in [0, T], \tag{43}$$

[36] See Schroder and Skiadas [1999] for the solution to the finite-horizon Epstein-Zin version of a Merton problem. The online appendix provides a heuristic derivation of the decision rules reported here, and for the equilibrium conditions that follow.

where ϕ_∞ is given below. Observe that ϕ_a is increasing in age and that $\phi_a \to \infty$ as a approaches T from below. This is how consumers end up spending all their wealth as they approach their terminal age T.

The dependence on age of these consumption-wealth ratios means that an equilibrium must, in general, depend on the distribution of wealth across different age cohorts of consumers alive at a given date. Here we will describe only how steady state equilibria are determined. It will no longer be the case that an unforeseen change in government policy immediately puts the economy in a new aggregate steady state. But it is possible for the government to augment an unforeseen change in fiscal targets with one-time age-specific proportional taxes on wealth and transfers of wealth to immediately implement the new steady state distribution of wealth across age cohorts.[37]

Given a risk-free rate equal to $r = g + x - \xi\varsigma^2\psi$, the parameter ϕ_∞ of the decision rule (43) and the resulting individual consumption growth rate g_y are determined by two conditions that are completely analogous to the conditions (27)-(28) for the $T = \infty$ economy,

$$g_y = g + \delta + x - \xi\varsigma^2\psi(1-\psi) - (\omega + \phi_\infty)\,, \tag{44}$$

$$\phi_\infty = \rho + \delta - \left(1 - \frac{1}{\varepsilon}\right)\left(g_y - \frac{1}{2}\xi\varsigma^2\psi^2\right). \tag{45}$$

Given the decision rules ψ and ϕ_a, individual consumer wealth is no longer a geometric Brownian motion. Its drift decreases with age, and very rapidly as a approaches T. But all consumers alive at a given point in time face the same expected returns and the same uncertainty. Because of this, the consumption $C_{j,t}$ of consumer j alive at time t again follows $\mathrm{d}C_{j,t} = C_{j,t}\left(g_y\mathrm{d}t + \psi\varsigma\mathrm{d}Z_{j,t}\right)$ conditional on survival.[38] At time t, aggregate consumption of the cohort born at date $t-a$ is then $C_{y,t}e^{-(g+\delta-g_y)a}$. In a steady state, $[C_t, C_{y,t}] = [C, C_y]e^{gt}$, and accounting for births and deaths shows that $C_y/C = ((1-e^{-\delta T})/\delta)/((1-e^{-(g+\delta-g_y)T})/(g+\delta-g_y))$.

Wealth at time t of a consumer j born at $t-a$ can be inferred from $(1+\tau)C_{j,t}/\phi_a$. This can be used to calculate aggregate steady state wealth and infer the aggregate consumption-wealth ratio. This yields

$$\frac{E}{W} = \phi_\infty\left(1 - e^{-\phi_\infty T}\left(\frac{1-e^{-(g+\delta-g_y)T}}{g+\delta-g_y}\right)^{-1}\frac{1-e^{-(g+\delta-(\phi_\infty+g_y))T}}{g+\delta-(\phi_\infty+g_y)}\right)^{-1}. \tag{46}$$

[37]To emphasize: the distribution of wealth within an age cohort still does not matter for determining the equilibrium.

[38]Since $\phi_a = E_{j,t+a}/W_{j,t+a}$ is determininstic, Ito's lemma implies that $\mathrm{d}E_{j,t+a} = W_{j,t+a}\mathrm{d}\phi_a + \phi_a\mathrm{d}W_{j,t+a}$. Together, (43) and the Merton wealth dynamics imply that the drift of individual consumption is constant.

Even though ϕ_∞ could be negative in a finite-T economy, the E/W implied by (46) is strictly positive by construction—it is a ratio of positive aggregate consumption and positive aggregate wealth. If $\phi_\infty > 0$ and $g_y < g + \delta$, as would be the case in the $T = \infty$ economy, then E/W converges to ϕ_∞ as T becomes large.

6.2 The Finite-T Equilibrium Conditions

The risky market clearing condition (24) for the $T = \infty$ economy still applies here, for finite T. But the risk-free market clearing condition (25) changes because the aggregate present value of the labor income of consumers alive at a given point in time has to account for their ages. A straightforward calculation gives

$$1 - \psi = \frac{\alpha}{\delta + r - g}\left(1 - \frac{1 - e^{-(r-g)T}}{r - g}\frac{\delta e^{-\delta T}}{1 - e^{-\delta T}}\right)\frac{1+\gamma}{1+\tau}\frac{E}{W} + \frac{D/P}{W}. \tag{47}$$

The conditions for a balanced growth path can then be obtained from (23)-(28) by replacing (25) with (47), replacing (27)-(28) with (44)-(45), and adding the new condition (46). The additional variable is the parameter ϕ_∞ of the age-dependent consumption-wealth ratio ϕ_a. Its sign is unrestricted because ϕ_a is automatically positive, as is the aggregate consumption-wealth ratio (46). As before, x has to be positive, and $r - g$ has to have the sign of the primary surplus. There is no requirement that $\delta + r - g$ is positive, because the present value of anyone's labor income is automatically finite for any $r - g$.

6.3 Large-T Convergence

Given an equilibrium in the $T = \infty$ economy, it is rather straightforward to pick fiscal targets for a large but finite T economy that generate an equilibrium close to that of the $T = \infty$ economy.

Fix some $\psi \in (0, 1)$ and $r - g > -\delta$ that characterize an equilibrium in the $T = \infty$ economy, given some fiscal targets τ and σ. This implies an $x > 0$, an aggregate consumption-wealth ratio $E/W > 0$, as well as g_y and g that satisfy $g_y < g + \delta$. To construct fiscal targets and an equilibrium for the finite-T economy, define $\phi_\infty = E/W$ and take ψ, $r - g$, x, g and g_y for the finite-T economy to be same as in the $T = \infty$ economy. By construction, this means that (44)-(45) holds. Furthermore, (46) implies that one can take T large enough so that E/W is arbitrarily close to ϕ_∞ in the finite-T economy. One can then use the risky market clearing condition (24) to construct a τ_T for the finite-T economy. Since E/W converges to its $T = \infty$ counterpart ϕ_∞, this τ_T converges to τ. The risk-free market clearing condition (47) can then be used to back out a σ_T. It will also converge to σ because the aggregate present value of labor income in the finite-T economy converges to its $T = \infty$ counterpart.

Proposition 13 *Fix some $\omega \geq 0$ and fiscal targets $\tau > -1$ and $\sigma > 0$. Suppose the $T = \infty$ economy has a well-defined equilibrium balanced growth path characterized by some ψ and $r - g$. For all T large enough, it is possible to find $\tau_T > -1$ and $\sigma_T > 0$ so that these ψ and $r - g$ are also part of an equilibrium balanced growth path in the finite-T economy. The resulting sequence of fiscal targets satisfies $(\tau_T, \sigma_T) \to (\tau, \sigma)$.*

For simplicity, the case $\sigma = 0$ is ruled out in this proposition to avoid complications that could arise from our assumption that transfers to newborn consumers have to be non-negative. With that caveat, this proposition applies to all $T = \infty$ economies that have an equilibrium, including the ones for which there is no upper bound on utility.

In the finite-T economy, the utility at time t of a consumer j born at time $t - a$ can be written as

$$U_{j,a} = C_t \times \frac{g + \delta - g_y}{\delta} \frac{1 - e^{-\delta T}}{1 - e^{-(g+\delta-g_y)T}} \left(\frac{\phi_a}{\rho + \delta} \right)^{-1/(1-1/\varepsilon)} M_{j,a} e^{(g_y - g)a}, \tag{48}$$

where ϕ_a is defined in (43) and $M_{j,a}$ is an individual-specific positive Brownian martingale with diffusion coefficient $\varsigma \psi M_{j,a}$ and initial value $M_{j,0} = 1$. For a newborn consumer at time t, this reduces to $C_t(C_y/C)(\phi_0/(\rho + \delta))^{-1/(1-1/\varepsilon)}$. Aggregate consumption is $C_t = (xK_t)^{1-\alpha} L^{\alpha}/(1 + \gamma)$. For the finite-$T$ equilibria constructed in the proof of Proposition 13, the x, g, and g_y, as well as the trajectory of K_t and the $\{M_{j,a}\}_{a \in [0,T]}$ are identical to what they are in the corresponding $T = \infty$ economy. For any age interval $[0, A] \subset [0, T)$, it then follows that the date-t utilities $\{U_{j,a}\}_{a \in [0,A]}$ converge to the corresponding utilities for the $T = \infty$ economy. In this sense, finite-T utilities also converge to their $T = \infty$ counterparts.

As already noted, it is possible for the finite-T economy to immediately switch to a new balanced growth path following an unforeseen change in fiscal targets, provided that such a change is accompanied by age-dependent taxes and transfers that put the distribution of wealth across age cohorts into its new steady state. Such a redistribution of wealth can also be implemented in the $T = \infty$ economy, and then (48) can be used to evaluate the welfare consequences for both $T < \infty$ and $T = \infty$. But for the $T = \infty$ economy, this results in a policy experiment that differs from the changes in τ and σ only that we have considered throughout. We leave the transitional dynamics in a finite-T economy of changes in τ and σ only to future work.

6.4 A Quantitative Example

When an economy is sufficiently productive, we know from Proposition 7 that large Pareto improvements will result from large transfers to newborn consumers, combined, if necessary, with a positive wealth tax. Large consumption taxes are not needed. Although it is not necessarily the case that the effect on utilities of increasing σ is monotone, it is certainly possible to construct robust examples (as in Figure 5) in which even small increases in these transfers are Pareto improving.

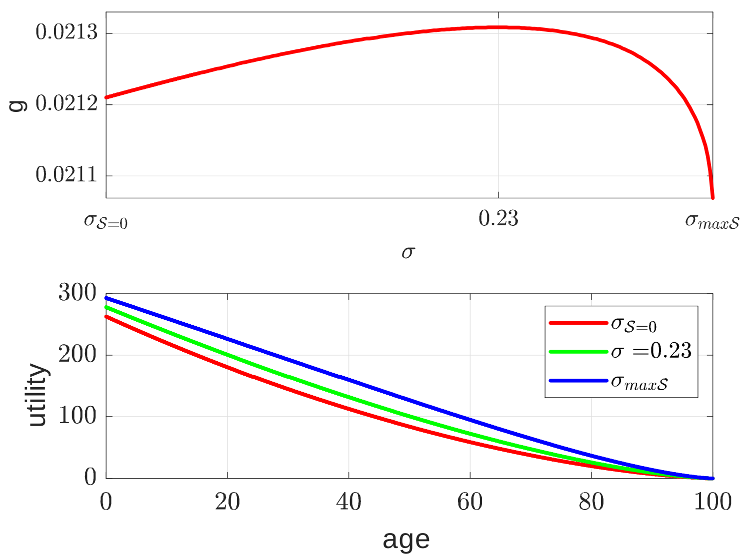

Figure 9 *Pareto improvements for* $T = 100$.

Here we add to this an example showing that increases in transfers to newborn consumers can be Pareto improving even in an economy in which consumers are finitely lived and do not care about their descendants. In the example, we consider unforeseen increases in σ that are accompanied by one-time age-dependent proportional wealth taxes and transfers at the time a new policy is implemented, in such a way that the distribution of wealth across a cohorts immediately jumps to its new steady state. In a steady state, the aggregate consumption at time t of consumers born at date $t - a$ is $C_{y,t}e^{-(g+\delta-g_y)a}$, and the size of this cohort is $e^{-\delta a}/(1 - e^{-\delta T})$, implying that the stationary distribution of per-capita consumption across cohorts of ages $a \in [0, T]$ has a density that scales with $e^{-(g-g_y)a}$. A new policy implies a new steady state value for $g - g_y$ and ϕ_a, and therefore for the distribution of wealth across cohorts as well. Age-dependent wealth taxes and transfers leave the within-cohort wealth distributions unaffected. The overall distribution of wealth will be in its new steady state only after T units of time, when the last cohort that lived through the unforeseen policy change leaves the scene.

In such a setting, the effects of an unforeseen one-time increase in σ are implied by the balanced growth conditions for the finite-T economy and (48). For individual consumers, the $M_{j,a}$ are unaffected, while the other factors in (48) jump upon the arrival of the new policy. Figure 9 displays a scenario in which increases in σ can lead to Pareto improvements. Over a bounded range of σ, the entire curve $\{U_{j,a}\}_{a\in[0,A]}$ shifts up with increases in σ, and the growth rate g increases as well. The increase in g ensures that C_s/C_t increases for all $s > t$, and hence that all future cohorts of consumers also gain from the increased transfers to newborn consumers. In this example, $T = 100$ and $\delta = 0.005$, resulting in an average age of about 46 years, and an average life span of almost 79 years. The intertemporal elasticity of substitution is large, $\varepsilon = 3$, and the economy is productive enough that its $T = \infty$ counterpart does not have an equilibrium when the wealth tax is zero.[39]

7 Conclusion

In an economy in which consumers are subject to uninsurable idiosyncratic long-run risk, the government may be able to run permanent primary deficits. How much the government can borrow very much depends on how it uses use the proceeds. If the government uses its deficits to make transfers to newborn consumers, then there may not be a bound on government borrowing. Specifically, if preferences imply an intertemporal elasticity of substitution greater than 1, and the economy is sufficiently productive, then there is no upper bound on the Pareto improvements that large transfers to newborn consumers can generate.

These unbounded Pareto improvements, implemented using a simple decentralization and very simple government policies, only arise in an economy in which a logically omniscient central planner in full control of everything in the economy could deliver infinite utility to everyone. Through the lens of more elaborate versions of our economy, it may well be that the empirical evidence, generated by a mostly decentralized market economy, leads to parameters for which large primary deficits are indeed Pareto improving. To rule these parameters out a priori because an imagined central planner could deliver bliss hard-wires a pessimistic view of how productive the underlying economy is and simply assumes away that large deficits can be Pareto improving via the mechanism we describe. And we have shown that these parameters also imply substantial Pareto improvements when households have a finite horizon and therefore bounded utility.

There are, of course, important caveats. A first important caveat, already hinted at in

[39]Specifically, $\mu = 0.09$. The remaining parameters are given by $\rho = 0.005$, $\xi = 7.5$, $\varsigma = 0.2$, and $\alpha = 0.6$. The fiscal targets satisfy $(1 + \gamma)/(1 + \tau) = 0.9$ and $\omega = 0$.

the introduction, is the Kareken-Wallace indeterminacy that applies when governments run permanent primary deficits. If the government can raise a lot of revenue by issuing securities that are not backed by future taxes, then there is the possibility that long-lived private-sector organizations emerge and compete with the government for these revenues.

A second important caveat is that we have taken labor supply to be inelastic. This means that we have abstracted from potentially important adverse effects of large transfers on output. Our model can easily be adapted to allow for consumers who can choose between more or less demanding careers only at the beginning of life. Even though this makes the supply of labor elastic, unbounded Pareto improvements are still a possibility. What happens when early career choices do not impose such tight restrictions on labor supply requires further study.

A third important caveat is that our model does not explain why markets are incomplete. It is very likely that risk sharing arrangements in actual economies are more sophisticated than we have assumed them to be. The gains from large government deficits we have described may be overstated once the effects of these deficits on private-sector risk-sharing arrangements are taken into account.

A fourth important caveat is that we have assumed that government policy never changes. Government securities may not be essentially risk free if political risk is important. An unforeseen transfer of newly issued government securities to a subset of the consumers alive at the time of the transfer causes a sudden inflation. This hurts all consumers who do not receive the transfer. The scope for such redistributions is large when the steady state supply of government securities is large. If consumers come to assign a positive probability to such events, then this will reduce their willingness to hold government securities, and hence the ability of the government to run persistent primary deficits.

We have already shown by example that small transfers to newborn consumers can be Pareto improving when the economy is sufficiently productive that unbounded Pareto improvements are possible. A richer model that incorporates the caveats we have just described would be needed to estimate how large the Pareto improvements within reach of government policy really are.

A Proof of Proposition 5

Write the risky and risk-free market clearing conditions as

$$\left(\xi\varsigma^2\psi + r - g\right)\psi = (1-\alpha) \times \frac{E/W}{1+\sigma}, \qquad (\delta + r - g)(1-\psi) = \alpha \times \frac{E/W}{1+\sigma},$$

respectively. The remaining equilibrium conditions (26)-(28) can be summarized by

$$\begin{aligned} 1 - \left(1 - \frac{1}{\varepsilon}\right)\frac{(1-\alpha)\mu + \delta - \frac{1}{2}\xi\varsigma^2}{\rho+\delta} \quad &= \quad \frac{1}{\rho+\delta}\frac{1+\sigma}{\varepsilon}\frac{E/W}{1+\sigma} \\ &+ \left(1-\frac{1}{\varepsilon}\right)\frac{\alpha\left(\xi\varsigma^2\psi - \delta + \delta + r - g\right) + \frac{1}{2}\xi\varsigma^2(1-\psi)^2}{\rho+\delta}. \end{aligned} \tag{49}$$

The side conditions are $\psi \in (0,1)$, $E/W > 0$, and $\delta + r - g > 0$. Because of these side conditions, the two market clearing conditions force $\psi = 1-\alpha = 1/(\xi\varsigma^2/\delta)$ if it so happens that $1 = (1-\alpha)\xi\varsigma^2/\delta$. This in turn implies $(E/W)/(1+\sigma) = \delta + r - g$, and then the right-hand side of (49) becomes linear in $\delta + r - g$. Varying $\delta + r - g > 0$ then proves the result. In what follows, take $1 \neq (1-\alpha)\xi\varsigma^2/\delta$.

Adding up the two market clearing conditions gives

$$\frac{E/W}{1+\sigma} = \delta + r - g - \left(\delta - \xi\varsigma^2\psi\right)\psi, \tag{50}$$

and eliminating E/W from the two market clearing conditions yields

$$\delta + r - g = \frac{\alpha\left(\delta - \xi\varsigma^2\psi\right)\psi}{\psi - (1-\alpha)}. \tag{51}$$

Using (50) to eliminate $(E/W)/(1+\sigma)$ from (49) gives

$$\begin{aligned} & 1 - \left(1 - \frac{1}{\varepsilon}\right)\frac{(1-\alpha)\mu + \delta - \frac{1}{2}\xi\varsigma^2}{\rho+\delta} \\ = \; & \left(\frac{1+\sigma}{\varepsilon} + \left(1-\frac{1}{\varepsilon}\right)\alpha\right)\frac{\delta+r-g}{\rho+\delta} - \left(\frac{1+\sigma}{\varepsilon} + \left(1-\frac{1}{\varepsilon}\right)\frac{1}{2}\right) \times \frac{\left(\delta - \xi\varsigma^2\psi\right)\psi}{\rho+\delta} \\ & + \left(1-\frac{1}{\varepsilon}\right)\left(\frac{(1-\alpha)\left(\delta-\xi\varsigma^2\psi\right)}{\rho+\delta} - \frac{\delta}{\rho+\delta}\left(1 - \frac{1}{2}\left(\psi + \frac{\xi\varsigma^2}{\delta}\right)\right)\right). \end{aligned}$$

Taking into account (51), the right-hand side of this equation only depends on ψ. Varying $\psi \in (0,1)$ over a domain that respects the side condition will trace out the set of μ for which the economy has an equilibrium.

A more concise way to write this backsolving equation is

$$1-\left(1-\frac{1}{\varepsilon}\right)\frac{(1-\alpha)\mu+\delta-\frac{1}{2}\xi\varsigma^2}{\rho+\delta}=\frac{\xi\varsigma^2}{\rho+\delta}\times f(\psi)$$

where

$$\begin{aligned} f(\psi) &= \left(\frac{1+\sigma}{\varepsilon}+\left(1-\frac{1}{\varepsilon}\right)\alpha\right)\frac{\alpha\,(k-\psi)\,\psi}{\psi-(1-\alpha)}-\left(\frac{1+\sigma}{\varepsilon}+\left(1-\frac{1}{\varepsilon}\right)\frac{1}{2}\right)(k-\psi)\,\psi \\ &\quad +\left(1-\frac{1}{\varepsilon}\right)\left((1-\alpha)\,(k-\psi)-\left(k-\frac{1}{2}\,(1+\psi k)\right)\right). \end{aligned}$$

and $k=\delta/(\xi\varsigma^2)\neq 1-\alpha$. The side condition $\delta+r-g>0$ together with (51) implies that $k-\psi$ and $\psi-(1-\alpha)$ must have the same sign. This means that the relevant domain $\mathcal{D}$ for $f(\psi)$ is $(1-\alpha,1)$ if $k\in(1,\infty)$, $(1-\alpha,k)$ if $k\in(1-\alpha,1)$, and $(k,1-\alpha)$ if $k\in(0,1-\alpha)$. By computing a second derivative, it is not difficult to verify that the right-hand side of (51) is convex in each of these scenarios. This right-hand side appears in the first term of $f(\psi)$, with a positive coefficient, and the remaining terms define a convex quadratic in ψ. So $f(\psi)$ is a convex function on each of the domains implied by the three possible scenarios.

Lemma A1 *Assume that $1-\alpha\neq k\in(0,\infty)$. The function $f:\mathcal{D}\to\mathbb{R}$ is convex, and*

(i) if $k\in(1,\infty)$ then $\mathcal{D}=(1-\alpha,1)$ and $\mathrm{D}f(1)<0$

(ii) if $k\in(1-\alpha,1)$ then $\mathcal{D}=(1-\alpha,k)$ and $\mathrm{D}f(k)<0$

(iii) if $k\in(0,1-\alpha)$ then $\mathcal{D}=(k,1-\alpha)$ and

$$\mathrm{D}f(k)>0 \text{ if and only if } \frac{1}{\varepsilon}>\frac{(1-\alpha)^2-k}{(1-\alpha)^2+\sigma k}$$

Furthermore, $f(1)=0$ and $f(\psi)\to\infty$, as $\psi\to 1-\alpha$ on the domain $\mathcal{D}$.

The derivative calculations that prove these results are collected in the online appendix.

The fact that $f(\psi)$ explodes near $1-\alpha$ means that $f(\psi)$ has no upper bound. If $k\in(1,\infty)$, then the convexity of $f(\psi)$ together with $f(1)=0$ and property (i) of Lemma A1 implies that the range of $f(\psi)$ is $(0,\infty)$. If $k\in(1-\alpha,1)$, then the convexity of $f(\psi)$ together with property (ii) implies a range $(f(k),\infty)$. And if $k\in(0,1-\alpha)$, then the convexity of $f(\psi)$ together with property (iii) again implies a range $(f(k),\infty)$, provided that the condition for $\mathrm{D}f(k)>0$ is met. Using $f(k)=-\left(1-\frac{1}{\varepsilon}\right)\frac{1}{2}\,(1-k)^2$ it is not difficult to check that this proves Proposition 5.

B Proof of Proposition 7

As long as budgets are not balanced, the equilibrium conditions can be summarized by the risky and risk-free market clearing conditions

$$\begin{aligned}\psi &= \frac{1-\alpha}{\xi\varsigma^2\psi+r-g}\frac{1+\gamma}{1+\tau}\frac{E}{W}, &(52)\\ 1-\psi &= \frac{\alpha}{\delta+r-g}\frac{1+\gamma}{1+\tau}\frac{E}{W}+\frac{1}{r-g}\left(\omega+\left(1-\frac{(1+\gamma)(1+\sigma)}{1+\tau}\right)\frac{E}{W}\right), &(53)\end{aligned}$$

together with the decision rule

$$\frac{1}{\varepsilon}\frac{E}{W}=\rho+\delta-\left(1-\frac{1}{\varepsilon}\right)\left((1-\alpha)\mu+\delta-\omega+\alpha\left(\xi\varsigma^2\psi+r-g\right)-\left(1-(1-\psi)^2\right)\times\frac{\xi\varsigma^2}{2}\right) \tag{54}$$

and the side conditions $\psi\in(0,1)$, $E/W>0$, and $\delta+r-g>0$. Furthermore, the second term on the right-hand side of (53) must be positive. To show existence for all $\sigma\geq 0$ large enough, one can vary $(\psi,r-g,E/W)$ subject to these side conditions, as well as (52) and (54), and then show that the range of $\sigma\geq 0$ implied by (53) is unbounded.

To this end, use (52) to eliminate $\xi\varsigma^2\psi+r-g$ from (54). This yields

$$\begin{aligned}&\left(\frac{1}{\varepsilon}+\left(1-\frac{1}{\varepsilon}\right)\frac{(1-\alpha)\alpha}{\psi}\frac{1+\gamma}{1+\tau}\right)\frac{E}{W}\\ =\;&\rho+\delta-\left(1-\frac{1}{\varepsilon}\right)\left((1-\alpha)\mu+\delta-\omega-\left(1-(1-\psi)^2\right)\times\frac{\xi\varsigma^2}{2}\right).\end{aligned} \tag{55}$$

Suppose that μ satisfies (33). Then (32) leads to $\psi_\infty\in(0,\min\{1,1/(\xi\varsigma^2/\delta)\}$, and hence $(r-g)_\infty=-\xi\varsigma^2\psi_\infty\in(-\delta,0)$. The definition of ψ_∞ says that the right-hand side of (55) is zero at $\psi=\psi_\infty$. Since $\varepsilon\neq 1$, and since $1-(1-\psi)^2$ is strictly increasing on $(0,1)$, varying $\psi\in(0,1)$ maps the right-hand side of (55) onto an open interval that contains 0. Suppose the factor multiplying E/W on the left-hand side of (55) is non-zero. Then it will be non-zero for all $\psi\in(0,1)$ in a small enough neighborhood of ψ_∞. One can then take $\psi>\psi_\infty$ or $\psi_\infty<\psi$ to obtain solutions for E/W from (55) that are positive, and that will converge to zero as $\psi\to\psi_\infty$. The $\xi\varsigma^2\psi+r-g$ implied by (52) are then also positive, and one can consider a small enough neighborhood of ψ_∞ to ensure that $\delta+r-g>0$ as well. Then use (53) to infer σ,

$$\frac{(1+\gamma)(1+\sigma)}{1+\tau}\frac{E}{W}=\omega+\frac{E}{W}-(r-g)(1-\psi)+\alpha\times\frac{r-g}{\delta+r-g}\frac{1+\gamma}{1+\tau}\frac{E}{W}.$$

Since $\omega - (r-g)_\infty(1-\psi) > 0$, the right-hand side will be positive and bounded away from zero for all $\psi \in (0,1)$ close enough to ψ_∞. Since $E/W \downarrow 0$ as $\psi \to \psi_\infty$, this means that the implied σ will become arbitrarily large as $\psi \to \psi_\infty$. It is not difficult to verify that second term on the right-hand side (53) will also be positive for ψ close enough to ψ_∞. This implies existence for all $\sigma \geq 0$ large enough.

This argument is predicated on the assumption that the coefficient multiplying E/W on the left-hand side of (55) is non-zero at ψ_∞. This is always true if $\varepsilon \in (1,\infty)$. But if $\varepsilon \in (0,1)$, then this assumption will fail for precisely one value of $\tau \in (-1,\infty)$. A tiny perturbation of τ can then restore the existence result.

C Proof of Proposition 8

The no-bubble equilibrium is defined by (34) and (35)with $\mathcal{S}/(r-g) = 0$. The risky market clearing condition (34) defines $r-g$ as a strictly decreasing function of $\psi \in (0,\infty)$, with a vertical asymptote at 0 and a large-ψ asymptote $-\xi\varsigma^2\psi$. The risk-free market clearing condition defines ψ as a strictly increasing function of $r-g > -\delta$ that tends to $-\infty$ as $r-g \downarrow -\delta$ and to 1 as $r-g \to \infty$. These curves intersect precisely once at some $\psi \in (0,1)$ and $r-g > -\delta$. The bound (37) follows from setting $D/(PE) = 0$ in (36) at $1+\gamma = 1+\tau$ and solving the quadratic for α subject to $1 - \alpha(\rho+\delta)/\delta > 0$. More generally, requiring $D/(PE) > 0$ in (36) is the same as requiring

$$k = \frac{(1-\alpha)/\alpha}{\xi\varsigma^2/\delta}, \qquad A = 1 - \alpha \times \frac{\rho+\delta}{\delta}\frac{1+\gamma}{1+\tau} > 0, \qquad A^2 > k\,(1-A)\,.$$

Replacing the second inequality by an equality gives a convex quadratic in A that has precisely one positive root, which is given by $-(k/2) + \sqrt{(k/2)^2 + k}$. It follows that $D/(PE) > 0$ is the same as requiring A to be to the right of this positive root. This can also be written as (38).

D Proof of Proposition 9

We already know that the economy has at least one equilibrium for every $\mathcal{S} \geq 0$. Focus therefore on equilibria with $\mathcal{S} < 0$. Using (34) to eliminate $(1+\gamma)/(1+\tau)$ from (35) gives

$$\mathcal{S} = \frac{r-g}{\rho+\delta}\left(1 - \psi - \frac{\alpha}{1-\alpha}\frac{(\xi\varsigma^2\psi + r - g)\psi}{\delta + r - g}\right).$$

We want to find the range of $\mathcal{S} < 0$ implied by varying $\psi \in (0,1)$ and $r-g<0$ subject to $\xi\varsigma^2\psi + r - g > 0$, and $\delta + r - g > 0$. Holding fixed some $r - g \in (-\min\{\delta, \xi\varsigma^2\}, 0)$, the above expression for $-\mathcal{S}$ is decreasing in $\psi \in (-(r-g)/(\xi\varsigma^2), 1) \subset (0,1)$. So $-\mathcal{S}$ approaches its supremum $-(r-g)(1-\psi)/(\rho+\delta)$ as $\psi \downarrow -(r-g)/(\xi\varsigma^2)$. This implies the upper bound

$$-\mathcal{S} < \frac{\delta}{\rho+\delta}\frac{\xi\varsigma^2}{\delta}\left(-\frac{r-g}{\xi\varsigma^2}\right)\left(1+\frac{r-g}{\xi\varsigma^2}\right).$$

Maximizing this upper bound over $r - g \in [-\min\{\delta, \xi\varsigma^2\}, 0]$ then gives $-(r-g)/\delta = \min\{1, (\xi\varsigma^2/\delta)/2\}$. The resulting maximum is the upper bound given in the proposition.

By letting $-(r-g)/\delta$ vary throughout $(0, \min\{1, (\xi\varsigma^2/\delta)/2\})$ and taking $\xi\varsigma^2\psi + r - g > 0$ close enough to zero, one can trace out all possible values of $-\mathcal{S}$ between 0 and this upper bound. Given any such $(\psi, r-g)$, the implicit $(1+\gamma)/(1+\tau)$ follows from (34), and then taking $\omega = 0$ and backing out σ from the definition of $\mathcal{S}$ completes the construction of an equilibrium. Because $\xi\varsigma^2\psi + r - g > 0$ can be taken to be arbitrarily close to zero, the risky market clearing condition (34) means that $(1+\tau)/(1+\gamma)$ can be made arbitrarily large, and then the definition of $\mathcal{S}$ together with $\omega = 0$ ensures that $\sigma > 0$.

E Proof of Proposition 10

Define $B = (\alpha+\theta)/(1+\theta)$. The risky and risk-free market clearing conditions are then

$$\begin{aligned}
\frac{\psi}{\rho+\delta+\omega} &= \frac{1-B}{\xi\varsigma^2\psi + r - g} \times (1-\mathcal{S}_*), \\
\frac{1-\psi}{\rho+\delta+\omega} &= \frac{B}{\delta + r - g} \times (1-\mathcal{S}_*) + \frac{\mathcal{S}_*}{r-g},
\end{aligned}$$

Holding fixed $r - g < 0$ and $\mathcal{S}_* < 0$, this implies

$$\begin{aligned}
-\frac{\partial}{\partial B}\frac{\psi}{\rho+\delta+\omega} &= \frac{1-\mathcal{S}_*}{2\xi\varsigma^2\psi + r - g}, \\
-\frac{\partial}{\partial B}\frac{\psi}{\rho+\delta+\omega} &= \frac{1-\mathcal{S}_*}{\delta + r - g},
\end{aligned}$$

respectively. This means that the risky market clearing curve shifts less than the risk-free market clearing curve if and only if $2\xi\varsigma^2\psi > \delta$. Suppose this is not true. Then, adding up the two market clearing conditions and using $\xi\varsigma^2\psi + r - g \leq \frac{1}{2}\delta + r - g < \delta$, $\delta + r - g < \delta$, and $\mathcal{S}_*/(r-g) > 0$ gives $\delta/(\rho+\delta+\omega) > 1 - \mathcal{S}_*$, which contradicts $\mathcal{S}_* < 0$.

Since a reduction in θ implies an decrease in B, this means that the risky market clearing condition shifts to the right by less than the risk-free market clearing condition when $\mathcal{S}_*$ is held fixed by a reduction in τ. So there continues to be an equilibrium, and the fact that the risky market clearing condition shifts to the right by less also implies that the reduction in θ has made it possible to increase $-\mathcal{S}_*$. This argument applies as long as θ is positive, and hence $\theta = 0$ is necessary for $-\mathcal{S}_*$ to be maximal.

F Proof of Proposition 12

Since $W_y/W = (g + \delta - g_y)/\delta$ must be positive, and since it is only increasing in ψ when $\psi \in (0, 1/2)$, we can restrict attention to the domain

$$\mathcal{D} = \left\{(\psi, x) \in \left(0, \frac{1}{2}\right) \times (0, \infty) : \frac{x}{\rho+\delta} < 1 + \frac{\xi\varsigma^2}{\rho+\delta} \times (1-\psi)\psi\right\}.$$

On this domain, we have

$$\begin{bmatrix} \frac{\partial \ln(U)}{\partial \psi} & \frac{\partial \ln(U_y)}{\partial \psi} \\ \frac{\partial \ln(U)}{\partial x} & \frac{\partial \ln(U_y)}{\partial x} \end{bmatrix} = \frac{1}{\rho+\delta} \begin{bmatrix} -(1-\psi)\,\xi\varsigma^2 & \left(\frac{1-2\psi}{1+(1-\psi)\psi\times\frac{\xi\varsigma^2}{\rho+\delta}-\frac{x}{\rho+\delta}} - (1-\psi)\right)\xi\varsigma^2 \\ \frac{1-\alpha}{x/(\rho+\delta)} + \alpha & -\frac{1}{1+(1-\psi)\psi\times\frac{\xi\varsigma^2}{\rho+\delta}-\frac{x}{\rho+\delta}} + \frac{1-\alpha}{x/(\rho+\delta)} + \alpha \end{bmatrix}.$$

The region of interest is

$$\mathcal{P} = \left\{(\psi, x) \in \mathcal{D} : \frac{\partial U_y}{\partial \psi} \geq 0,\ -\frac{\partial U/\partial x}{\partial U/\partial \psi} \geq -\frac{\partial U_y/\partial x}{\partial U_y/\partial \psi}\right\}.$$

It is not difficult to verify that anywhere in the domain $\mathcal{D}$,

$$\frac{\partial U_y}{\partial \psi} \geq 0 \Leftrightarrow (1-\psi)\,\psi \times \frac{\xi\varsigma^2}{\rho+\delta} + \frac{\psi}{1-\psi} \leq \frac{x}{\rho+\delta}, \tag{56}$$

Furthermore, given $\partial U_y/\partial \psi \geq 0$ and $(\psi, x) \in \mathcal{D}$,

$$-\frac{\partial U/\partial x}{\partial U/\partial \psi} \geq -\frac{\partial U_y/\partial x}{\partial U_y/\partial \psi} \Leftrightarrow \frac{x}{\rho+\delta} \leq 1 - \frac{\psi}{(1-\alpha)(1-\psi)+\alpha\psi}. \tag{57}$$

In both cases, the equivalences also hold for strict inequalities. The left-hand side of the inequality (56) is strictly increasing in $\psi \in (0, 1/2)$, and the slope of the right-hand side of the inequality (57) is $-\psi/((1-\alpha)(1-\psi)+\alpha\psi)^2$. So (57) defines the upper bound of the region $\mathcal{P}$ and (56) the lower bound. At $\psi = 1/2$, the lower bound exceeds 1, and the upper bound is zero. So there will be some $\psi \in (0, 1/2)$ where (56) and (57) both hold with equality. This corresponds to the (ψ, x) that maximizes U_y. An important fact about the lower bound (56) of $\mathcal{P}$ and the indifference curves of U is that the lower bound of $\mathcal{P}$ is steeper whenever the two meet. This follows from the fact that

$$\frac{1-\psi}{\frac{1-\alpha}{x/(\rho+\delta)}+\alpha}\frac{\xi\varsigma^2}{\rho+\delta} < (1-2\psi)\times\frac{\xi\varsigma^2}{\rho+\delta}+\frac{1}{(1-\psi)^2}$$

for all (ψ, x) in $\mathcal{P}$. This is true because dropping the second term on the right-hand side of this inequality gives the inequality (57). So an indifference curve of U cannot leave $\mathcal{P}$ as $\psi \in (0,1)$ goes down.

Take any $(\psi, x) \in \mathcal{P}$. Increasing x lowers g and therefore hurts generations that will be born far into the future. Lowering ψ by itself hurts the current newborn generation because $\partial \ln(U_y)/\partial\psi \geq 0$ in $\mathcal{P}$. Along an indifference curve for U,

$$\frac{\mathrm{d}\ln(U_y)}{\mathrm{d}\psi}=\frac{\partial\ln(U_y)}{\partial\psi}+\frac{\partial\ln(U_y)}{\partial x}\left(-\frac{\partial U/\partial\psi}{\partial U/\partial x}\right)=\frac{\partial\ln(U_y)}{\partial\psi}\left(1-\frac{-\frac{\partial U_y/\partial x}{\partial U_y/\partial\psi}}{-\frac{\partial U/\partial x}{\partial U/\partial\psi}}\right)\geq 0.$$

Therefore, lowering ψ along an indifference curve for U also hurts newborn consumers. Since the indifference curve for U stays inside $\mathcal{P}$ when ψ decreases, this rules out Pareto improvements starting from any $(\psi, x) \in \mathcal{P}$. Conversely, take some $(\psi, x) \in \mathcal{D}\backslash\mathcal{P}$. If (56) is violated, then lowering ψ by some amount is a Pareto improvement. If (56) holds but (57) is violated, then lowering ψ along the indifference curve for U is a Pareto improvement. So every allocation outside $\mathcal{P}$ can be improved upon.

References

[1] Abel, A.B. and S. Panageas, "Optimal Rollover of Government Bonds in a Dynamically Efficient Economy" Conference on Macroeconomics, Yale (2022).

[2] Aguiar, M.A., M. Amador, and C. Arellano, "Micro Risks and Pareto Improving Policies with Low Interest Rates," *NBER working paper 28996*, July 2021.

[3] Angeletos, G.-M., "Uninsured Idiosyncratic Investment Risk and Aggregate Saving," *Review of Economic Dynamics*, vol. 10, no. 1 (2007), 1–30.

[4] Atkeson, A. and A. Burstein, "Aggregate Implications of Innovation Policy," *Journal of Political Economy*, vol. 127 no. 6 (2019), 2625–2683.

[5] Aoki, K., T. Nakajima, and K. Nikolov, "Safe Asset Shortage and Asset Price Bubbles," *Journal of Mathematical Economics*, vol. 53 (2014), 164–174.

[6] Aoki, S., M. Nirei, "Zipf's Law, Pareto's Law, and the Evolution of Top Incomes in the United States," *American Economic Journal: Macroeconomics*, vol. 9, no. 3 (2017), 36-71.

[7] Bansal, R. and A. Yaron, "Risk for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *The Journal of Finance*, vol. 59, no. 4 (2004), 1386–1407.

[8] Barro, R.J., "r Minus g", *NBER working paper 28002*, October 2020.

[9] Benhabib, J. and A. Bisin, "The Distribution of Wealth: Intergenerational Transmission and Redistributive Policies", New York University (2007).

[10] Blanchard, O.J., "Debt, Deficits, and Finite Horizons," *Journal of Political Economy*, vol. 93, no. 2 (1985), 223–247.

[11] Blanchard, O.J., "Public Debt and Low Interest Rates," *American Economic Review*, vol. 109 (2019), 1197-1229.

[12] Blanchard, O.J. and P. Weil, "Dynamic Efficiency, the Riskless Rate, and Debt Ponzi Games under Uncertainty," *Advances in Macroeconomics*, vol. 1, no. 2 (2001).

[13] Brumm, J., X. Feng, L.J. Kotlikoff, and F. Kubler, "When Interest Rates Go Low, Should Public Debt Go High?," *NBER working paper 28952* (2021).

[14] Brumm, J., X. Feng, L.J. Kotlikoff, and F. Kubler, "Are Deficits Free?" *Journal of Public Economics*, vol. 208 (2022) 1–5.

[15] Brunnermeier, M.K., S.A. Merkel, and Y. Sannikov, "Debt as Safe Asset," *NBER working paper 29626* (2022).

[16] Boldrin, M. and D.K. Levine, "Quality Ladders, Competition and Endogenous Growth," working paper (2010).

[17] Diamond, P., "National Debt in a Neoclassical Growth Model," *American Economic Review*, vol. 55, no. 5 (1965), 1126-1150.

[18] Duffie, D. and L.G. Epstein, "Stochastic Differential Utility", *Econometrica*, vol. 60, no.2 (1992), 353-394.

[19] Epstein, L.G. and S.E. Zin, “Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework,” *Econometrica*, vol. 57 (1989), 937-969.

[20] Gabaix, X., J.-M. Lasry, P.-L. Lions, and B. Moll, “The Dynamics of Inequality,” *Econometrica*, vol. 84, no. 6 (2016), 2071-2111.

[21] Galor, O., “A Two-Sector Overlapping-Generations Model: A Global Characterization of the Dynamical System,” *Econometrica*, vol. 60, no. 6 (1992), 1351-1386.

[22] Greenwood, J. and B. Jovanovic, “Accounting for Growth,” in *New Developments in Productivity Analysis*, edited by C.R. Hulten, E.R. Dean, and M.J. Harper, NBER (2001), 179-222.

[23] Harrison, J.M. and D.M Kreps, “Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations,” *Quarterly Journal of Economics*, vol. 92, no. 2 (1978), 323–336.

[24] Hellwig, M., “Safe Assets, Risky Assets, and Dynamic Inefficiency in Overlapping-Generations Economies,” Discussion Papers of the Max Planck Institute for Research on Collective Goods, no. 10 (2021).

[25] Jones, L. and R. Manuelli, “Finite Lifetimes and Growth,” *Journal of Economic Theory*, vol. 58, no. 2 (1992), 171-197.

[26] Jones, L. and R. Manuelli, “Neoclassical Models of Endogenous Growth: The Effects of Fiscal Policy, Innovation and Fluctuations” *Chapter 1 in The Handbook of Economic Growth*, volume 1, part A (2005), 13–65.

[27] Kareken, J. and Neil Wallace, “On the Indeterminacy of Equilibrium Exchange Rates,” *Quarterly Journal of Economics*, vol. 96, no. 2 (1981), 207-222.

[28] Kocherlakota, N., “Public Debt Bubbles in Heterogeneous Agent Models with Tail Risk,” *International Economic Review*, forthcoming (2022).

[29] Luttmer, E.G.J., “Growth with a Fixed Factor,” Minnesota Economics Research Report (2012).

[30] Mehrotra, N. and D. Sergeyev, “Debt Sustainability in a Low Interest Rate World,” *Journal of Monetary Economics*, vol. 124S (2021), S1-S18.

[31] Miao,J., "Introduction to Economic Theory of Bubbles," *Journal of Mathematical Economics*, vol. 53 (2014), 130–136.

[32] Muller, W.J., III, and M. Woodford, "Determinacy of Equilibrium in Stationary Economies with Both Finite and Infinitely Lived Consumers," *Journal of Economic Theory*, vol. 46, no. 2 (1988), 255-290.

[33] Rebelo, S. "Long-Run Policy Analysis and Long-Run Growth", *Journal of Political Economy*, vol. 99, no. 3 (1991), 500–521.

[34] Reis, R., "The Constraint of Public Debt When $r < g$ but $g < m$," London School of Economics, March 2021.

[35] Santos, M. and M. Woodford, "Rational Asset Pricing Bubbles," *Econometrica*, vol. 65, no. 1 (1997), 19-57.

[36] Schorfheide, F., D. Song, and A. Yaron, "Identifying Long-Run Risks: A Bayesian Mixed-Frequency Approach," *Econometrica*, vol. 86, no. 2 (2018), 617–654.

[37] Schroder, M. and C. Skiadas, "Optimal Consumption and Portfolio Selection with Stochastic Differential Utility," *Journal of Economic Theory*, vol. 89, no. 1, (1999), 68-126.

[38] Toda, A.A., "Incomplete Market Dynamics and Cross-Sectional Distributions," *Journal of Economic Theory*, vol. 154 (2014), 310–348.

[39] Uzawa, H., "Optimum Technical Change in An Aggregative Model of Economic Growth", *International Economic Review*, vol. 6, no 1 (1965), 18–31.

[40] Weil, P., "Overlapping Families of Infinitely-lived Agents," *Journal of Public Economics*, vol. 38, no. 2 (1989), 183–198.

[41] Yaari, M., "Uncertain Lifetime, Life Insurance, and the Theory of the Consumer," *Review of Economics and Studies*, vol. 32, no. 2 (1965), 137–150.

[42] Yogo, M., "Estimating the Elasticity of Intertemporal Substitution When Instruments Are Weak," *Review of Economics and Statistics*, vol. 86, no. 3 (2004), 797–810.