A Note On Wiener-Kolmogorov Prediction Formulas for Rational Expectations Models

Lars Peter Hansen

Carnegie-Mellon University

Thomas J. Sargent

University of Minnesota and
Federal Reserve Bank of Minneapolis

Research Department Staff Report 69
Federal Reserve Bank of Minneapolis

September 1981

ABSTRACT

A prediction formula for geometrically declining sums of future forcing variables is derived for models in which the forcing variables are generated by a vector autoregressive-moving average process. This formula is useful in deducing and characterizing cross-equation restrictions implied by linear rational expectations models.

In linear rational expectations models it is commonplace to encounter infinite geometrically declining sums of expected values of future variables. For example, in models studied by Sargent (1977, 1978a, 1978b), Salemi and Sargent (1978), and Hansen and Sargent (1980, 1981b), the actions of economic agents depend on geometrically declining sums of expected values of variables that agents view as beyond their control. These sums also occur in rational expectations competitive equilibrium models whose equilibrium time paths can be obtained by solving a fictitious quadratic objective function - linear constraint social planning problem. There the decision rule of the fictitious social planner oftentimes embeds the solution of an optimal forecasting problem involving geometrically declining sums of forecasts of variables that cannot collectively be influenced by the competitive agents [see Sargent (1981), Hansen and Sargent (1981a), and Eichenbaum (1981)]. Thus from the standpoint of rational expectations model derivation and parameter estimation, it is convenient to have an explicit solution to the prediction problem

$$y_t = \sum_{j=0}^{\infty} \lambda^j [x_{t+j} | \Omega_t]$$

where x_t is a p x 1 vector of variables observed by economic agents, E is the mathematical expectations operator, Ω_t is an information set that is a subset of that used by rational economic agents, 1/ and λ is a scaler parameter such that $|\lambda| < 1$. In dynamic optimizing models, the parameter λ typically depends on the criterion function parameters of the optimization problem assumed to be solved by private agents or by the fictitious social planner. Previously [Hansen and Sargent (1980)], we have demonstrated how to use Wiener-Kolmogorov least squares prediction theory in obtaining a solution for y_t under special assumptions about x_t and the information set Ω_t. In this note we extend those results.

To begin, we specify Ω_t more precisely and suppose that there is a q dimensional stochastic process z that generates Ω_t in the sense that $\Omega_t = \{z_s : -\infty < s \leq t\}$. In other words, current and past values of z summarize all of the information in Ω_t. As a special case, x_t can be a subvector of z_t. Next we assume that the joint stochasitc process (z', x')' is covariance stationary and linearly indeterministic and has mean zero. 2/ Furthermore, we assume that linear least squares predictors coincide with conditional expectations. Applying Wold's Decomposition Theorem to the z process, we know that z_t can be represented as

(2) $$z_t = \alpha(L)u_t$$

where $\alpha(L) = \alpha_0 + \alpha_1,L + \ldots$ is an infinite order matrix lag operator that satisfies

$$\sum_{j=0}^{\infty} \text{trace}\ \alpha_j\alpha_j' < +\infty,$$

u_t is orthogonal to Ω_{t-1}, and $u_t \in \Omega_t$. This provides us with an orthogonal decomposition of the information set Ω_t since $\Omega_t = \{u_s : -\infty < s \leq t\}$ and $Eu_tu_s = 0$ for $s \neq t$. The linear least squares forecast of x_t given Ω_t can be represented as

(3) $$\hat{E}[x_t|\Omega_t] = \beta(L)u_t$$

where $\hat{E}$ is the linear least squares projection operator. In the special case in which $x_t = z_t$, $\beta(L)$ and $\alpha(L)$ are equal. It is an implication of (3) that x_t satisfies

(4) $$x_t = \beta(L)u_t + v_t$$

where v_t is orthogonal to Ω_t.

We can obtain an infinite, geometrically declining sum of future x's by applying the operator $\frac{1}{1-\lambda L^{-1}}$ to x_t. Operating on both sides of (4) yields

$$(5) \qquad \left[\frac{1}{1-\lambda L^{-1}}\right] x_t = \left[\frac{\beta(L)}{1-\lambda L^{-1}}\right] u_t + \left[\frac{1}{1-\lambda L^{-1}}\right] v_t .$$

An iterated projection argument implies that $\hat{E}[v_s|\Omega_t] = 0$ for $s > t$, so that taking projections of both sides of (5) results in

$$y_t = \hat{E}\left[\left(\frac{1}{1-\lambda L^{-1}}\right)x_t \mid \Omega_t\right] = \hat{E}\left[\left(\frac{\beta(L)}{1-\lambda L^{-1}}\right) u_t \mid \Omega_t\right] .$$

Using the Wiener-Kolmogorov prediction formula we obtain

$$(6) \qquad y_t = \left[\frac{\beta(L)}{(1-\lambda L^{-1})}\right]_+ u_t$$

where $[\quad]_+$ is the annihilation operator that instructs us to ignore negative powers of L. At this point we can employ the Lemma in Appendix A of Hansen and Sargent (1980) to ascertain that

$$(7) \qquad \left[\frac{\beta(L)}{1-\lambda L^{-1}}\right]_+ = \frac{L\beta(L) - \lambda\beta(\lambda)}{L - \lambda} .$$

Both the numerator and denominator of (7) have the common factor $(L - \lambda)$ which can be divided out. [3/] The contribution of this note is to investigate the implications of (6) and (7) for a class of rational parameterizations of the infinite order matrix polynomial in the lag operator $\beta(L)$.

Suppose that z_t can be represented as

$$(8) \qquad A(L)z_t = B(L)u_t$$

where A(L) and B(L) are q x q finite order polynomials in the lag operator, det $A(\zeta)$ has its zeroes outside the unit circle $(|\zeta| = 1)$, and det $B(\zeta)$

does not have any zeroes inside the unit circle. In our previous papers [Hansen and Sargent (1980) and Hansen and Sargent (1981b)] we assumed that $B(L) = I$ or equivalently that z has a finite order autoregressive representation. We can invert the $A(L)$ operator and conclude that

$$A(L)^{-1}B(L) = \alpha(L)$$

where $\alpha(L)$ is the Wold Decomposition polynomial in the lag operator given in (2).

We also suppose that

$$\hat{E}[x_t|\Omega_t] = C(L)z_t \tag{9}$$

where $C(L)$ is a finite order p x q matrix polynomial in the lag operator. Specifications (8) and (9) imply that

$$\beta(L) = C(L)A(L)^{-1}B(L). \tag{10}$$

We shall use this specification of $\beta(L)$ together with relations (6) and (7) to ascertain a solution for y_t in terms of a finite number of z's and a finite number of u's.

Substituting (10) into (7) we deduce that

$$\left[\frac{\beta(L)}{1-\lambda L^{-1}}\right]_+ = \frac{LC(L)A(L)^{-1}B(L) - \lambda C(\lambda)A(\lambda)^{-1}B(\lambda)}{L-\lambda}\ . \tag{11}$$

The left-hand side of (11) can be split into two pieces:

$$\left[\frac{\beta(L)}{1-\lambda L^{-1}}\right]_+ = \left[\frac{LC(L) - \lambda C(\lambda)A(\lambda)^{-1}A(L)}{L-\lambda}\right] A(L)^{-1}B(L) + \lambda C(\lambda)A(\lambda)^{-1}\left[\frac{B(L) - B(\lambda)}{L-\lambda}\right]\ . \tag{12}$$

We let $B(L) = B_0 + B_1 L + \ldots + B_n L^n$ and for convenience we assume that

$$C(L) = C_0 + C_1 L + \ldots + C_m L^m$$

$$A(L) = A_0 + A_1 L + \ldots + A_{m+1} L^{m+1}.$$

The restriction on the orders of A(L) and C(L) can always be satisfied by defining some of the A_j or C_j matrices to be matrices of zeroes. To allow for the order of C and A to be arbitrarily related is straightforward but notationally cumbersome.

Next we perform polynomial division and determine that

$$\frac{LC(L) - \lambda C(\lambda)A(\lambda)^{-1}A(L)}{L - \lambda} = D_0 + D_1 L + \ldots + D_m L^m = D(L)$$

where

$$\begin{aligned} D_0 &= C_0 + C_1\lambda + \ldots + C_m\lambda^m - \lambda C(\lambda)A(\lambda)^{-1}(A_1 + A_2\lambda + \ldots + A_{m+1}\lambda^m) \\ D_1 &= C_1 + C_2\lambda + \ldots + C_{m-1}\lambda^{m-1} - \lambda C(\lambda)A(\lambda)^{-1}(A_2 + A_3\lambda + \ldots + A_{m+1}\lambda^{m-1}) \\ &\vdots \\ D_m &= C_m - \lambda C(\lambda)A(\lambda)^{-1}A_{m+1}. \end{aligned} \tag{13}$$

and that

$$\lambda C(\lambda)A(\lambda)^{-1}\frac{B(L) - B(\lambda)}{L - \lambda} = F_0 + F_1 L + \ldots + F_{n-1}L^{n-1} = F(L)$$

where

$$F_0 = \lambda C(\lambda)A(\lambda)^{-1}(B_1 + B_2\lambda + \ldots + B_n\lambda^{n-1})$$

$$F_1 = \lambda C(\lambda)A(\lambda)^{-1}(B_2 + B_3\lambda + \ldots + B_n\lambda^{n-1}$$

(14) $\vdots$

$$F_{n-1} = \lambda C(\lambda)A(\lambda)^{-1} B_n.$$

Substituting into (6) we see that

(15) $$y_t = D(L)A(L)^{-1}B(L)u_t + F(L)u_t$$

$$= D(L)z_t + F(L)u_t.$$

Equations (13) - (15) provide the solution to the optimal prediction problem of forecasting an infinite geometrically declining sum of expected future values of x. Both D(L) and F(L) are finite order lag polynomials where the coefficients are explicit functions of A_0, A_1, A_2, ..., A_{m+1}, B_0, B_1, ..., B_n, C_0, C_1, ..., C_m, and λ. Hence (15) expresses y_t as a function of the current value and m lagged values of z_t and the current value and n-1 lagged values of the one-step ahead forecast error in z_t. This extends results in our previous papers [Hansen and Sargent (1980) and Hansen and Sargent (1981b)]. There the F(L) polynomial was implicitly assumed to be zero. Formulas such as (13) - (15) are important in making econometric estimation of rational expectation models computationally tractable. In this particular case, they help make it practical to accommodate forcing variables with finite order vector autoregressive-moving average representations.

Footnotes

1/ Hansen and Sargent (1981b) discuss how to estimate parameters and test restrictions when Ω_t is a proper subset of the information set used by economic agents. As has been noted by Shiller (1972), by making Ω_t a proper subset one introduces a distrubance term that is orthogonal to elements in Ω_t.

2/ Hansen and Sargent (1980) show how predictions formulas derived under covariance stationarity can be applied to certain nonstationary processes with time invariant representations in which the exponential order of the process is less that $1/|\lambda|$.

3/ In the language of complex analysis the function $\frac{z\beta(z) - \lambda\beta(\lambda)}{z - \lambda}$ has a removable singularity at $z = \lambda$.

References

Eichenbaum, M. S., 1981, "A Rational Expectations Equilibrium Model of the Cyclical Behavior of Inventories and Employment," Carnegie-Mellon University.

Hansen, L. P. and T. J. Sargent, 1980, "Formulating and Estimating Dynamic Linear Rational Expectations Models," Journal of Economic Dynamics and Control, 2, 7-46.

Hansen, L. P. and T. J. Sargent, 1981a, "Linear Rational Expectations Models for Dynamically Interrelated Variables," in R. E. Lucas, Jr. and T. J. Sargent, eds., Rational Expectations and Econometric Practice (University of Minnesota Press: Minneapolis, MN).

Hansen, L. P. and T. J. Sargent, 1981b, "Instrumental Variables Procedures for Estimating Linear Rational Expectations Models," forthcoming in Journal of Monetary Economics.

Salemi, M. and T. J. Sargent, 1979, "The Demand for Money during Hyperinflations under Rational Expectations: II," International Economic Review, 20, 741-758.

Sargent, T. J., 1977, "The Demand for Money during Hyperinflations under Rational Expectations: I," International Economic Review, 18, 59-82.

Sargent, T. J., 1978a, "Rational Expectations, Econometric Exogeneity, and Consumption," Journal of Political Economy, 86, 673-700.

Sargent, T. J., 1978b, "Estimation of Dynamic Labor Demand Schedules under Rational Expectations," Journal of Political Economy, 86, 1009-1044.

Sargent, T. J., 1981, "Interpreting Economic Time Series," Journal of Political Economy, 89, 213-248.

Shiller, R., 1972, "Rational Expectations and the Structure of Interest Rates," Ph.D. dissertation (Massachusetts Institute of Technology, Cambridge, MA).