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Factor Saving Innovation*

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ABSTRACT

We study a simple model of factor saving technological innovation in a concave framework. Capital can be used either to reproduce itself or, at additional cost, to produce a higher quality of capital that requires less labor input. If higher quality capital can be produced quickly, we get a model of exogenous balanced growth as a special case. If, however, higher quality capital can be produced slowly, we get a model of endogenous growth in which the growth rate of the economy and the rate of adoption of new technologies are determined by preferences, technology, and initial conditions. Moreover, in the latter case, the process of growth is necessarily uneven, exhibiting a natural cycle with alternating periods of high and low growth. Growth paths and technological innovations also exhibit dependence upon initial conditions. The model provides a step toward a theory of endogenous innovation under conditions of perfect competition.

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1 Introduction

We contribute to the debate on the endogeneity of aggregate technological progress by introducing a concave model of innovation with three properties. Technological innovations are factor saving; implementable only in discrete lumps; and endogenous, depending on people's decisions. We find in such circumstances that growth can be path-dependent and uneven over time.

In our model, technological innovation takes place through the adoption of new activities that make use of new qualities of capital. There is a fixed factor, and more advanced activities are superior in the sense that they make less use of this fixed factor. This means that technological innovation is *biased* or *factor saving*. For concreteness, we refer to the fixed factor as labor. In this setting, investment provokes both *capital widening*, meaning that the total stock of capital grows larger, and *capital deepening*, meaning that the quality of the capital stock is improved. In fact, because of the fixed labor supply, capital deepening is necessary for capital widening.

Without moving to higher qualities of capital that use less labor per unit of output, there is no reason to build more capital. Introducing new capital goods is costly because, for given inputs, the capital deepening technology yields fewer units of output than the capital widening technology. Hence, regardless of how long the capital deepening activity has been available, it will be used only when relative prices make it profitable to do so, that is, when the labor supply becomes a binding constraint. In this sense, technological innovation is fully endogenous.

Our model is one of perfect competition with constant returns to scale. A variety of arguments have been advanced as to why growth models with increasing returns are superior to those with diminishing or constant returns. From a theoretical standpoint, the endogenous versus exogenous nature of economic growth is the principal argument. Romer (1994, p. 12), for example, says that the fact that "technological advance comes from things that people do" and is not merely "a function of elapsed calendar time" argues against concave models of "exogenous" technological change. In this interpretation, endogeneity means that technological innovations should come from "things people do."

In our setting, technical advance clearly comes about because it is profitable for innovators to introduce new technologies into production. This should clarify that growth can be as endogenous in a concave setting as it is when there are externalities, increasing returns, monopoly power, and so forth. To the best of our knowledge, this is the first time such a result is clearly proven.

Our equilibrium path distinguishes between growth due to the accumulation of factors and the introduction of new factors and activities, which we refer to as *technological innovation*. To be concrete, we will propose that the growth rate and the rate of technological innovation are endogenous if they are affected in a nontrivial way by changes in the rate of intertemporal substitution in consumption. Notice that in the Solow growth model, neither the growth rate nor the rate of technological innovation is endogenous in this sense. In Jones and Manuelli's (1990) and Rebelo's (1991) AK models, the growth rate is endogenous, but the rate of technological innovation is not. On the other hand, in models with increasing returns, such as those of Lucas (1988) and Romer (1990), not only is the growth rate endogenous, but so is the rate of technological innovation.

The endogeneity of growth in our model depends on how rapidly it is possible to produce higher quality capital. If capital of higher quality can be produced quickly, we get a model identical in essence to the exogenous growth model of Solow. A new quality of capital is introduced every period, and the economy grows at a fixed rate independent of the subjective discount factor and other preference or technology parameters. If new quality capital can only be produced slowly, the situation changes drastically. Both the growth rate and the rate of technological innovation are fully endogenous and depend on the subjective discount factor and other preference parameters.

In our model, technological innovations clearly come from things that people do. In fact, contrary to models where externalities carry the day, technological innovations here come from things that people consciously choose to do. They introduce new technologies in those periods when they are needed to relax the labor constraint, and they do not introduce new technologies in periods in which such need is absent. Note that we do not attempt to model the reason that technologies become available. We assume that technologies are available or become available for reasons exogenous to the model. Our theory of technological innovation is a theory about why those technologies are actually introduced. A theory of why new technologies become available in a concave world is presented in Boldrin and Levine (2000).

As we mentioned at the outset, the most striking feature of our model is that equilibria are path-dependent and do not exhibit a constant growth rate. Growth follows a natural cycle in which gradual upward increases in consumption are interrupted by periodic growth recessions in which consumption remains flat. These periods of creative destruction are those in which a shift to a new technological paradigm first takes place. The existence of growth cycles can be extended to models with many goods, sectors, and factors of production, as long as natural resources are essential in production and innovation has an impact on many sectors. While it can be argued that many innovations are cumulative in nature, so that the introduction of a new technology has only a slight effect on the economy, innumerable important innovations, such as the use of personal computers, the introduction of electrical power, and the advent of new business methods, are of general purpose and can be expected to have a substantial impact across many sectors. In the presence of a sequence of such large innovations, the process of growth will be uneven, with spurts of growth as the new technology is exploited and periods of relative stagnation while the new capital good is accumulated for the next growth spurt. Note that during growth recessions, the economy remains at full employment; unemployment occurs only in the case of stagnation when the capital stock becomes too small to employ all existing labor.

In addition to endogenous growth and a natural growth cycle, our economy exhibits path-dependence, meaning that the long-run growth rate of the economy can depend on the initial stock of capital. Indeed, a small change in initial capital can make the difference between long-run innovation and growth and long-run stagnation and decline. In particular, it is possible for the economy to grow in the short run, with new technologies and increased consumption per capita, yet fall back into stagnation in the long run with declining consumption, rising unemployment, and only the lowest possible quality technology employed. Again, to the best of our knowledge, we are not aware of any dynamic model exhibiting this rather frequently observed pattern.

2 The Model

We consider an infinite horizon economy, $t = 0, 1, 2, \dots$, with a continuum of homogeneous consumers. Consumers value consumption $c_t \in \Re_+$. The period utility function $u(c_t)$ is bounded below, continuously differentiable, strictly increasing, and strictly concave. It satisfies the Inada conditions $\lim_{c \to 0} u'(c) = +\infty$ and $\lim_{c \to +\infty} u'(c) = 0$. Total lifetime utility is given by

$U(c) = \sum_{t=0}^{\infty} \delta^t u(c_t)$, where $0 \leq \delta < 1$ is the common subjective discount factor. Let $\mu = \sup\{\widetilde{\mu} \| \sum_{t=0}^{\infty} \delta^t u(\widetilde{\mu}^t) < \infty\}$ be the supremum of growth rates for which utility remains finite. Notice that $\mu \geq 1/\delta$.

Consumption is produced by activities that use labor and capital as inputs. In addition, capital is produced from capital, and labor reproduces itself. Capital comes in an infinite sequence of qualities, indexed by $i = 0, 1, \ldots$.

We write an input vector as $z = (\kappa, \ell)$, where κ is an infinite vector of capital stocks of different qualities and ℓ is a scalar denoting labor. The period input space Z consists of the set of sequences $(z_1, z_2, \ldots, z_n, \ldots) \geq 0$ for which $z_n = 0$ for all but finitely many n. Note that the technology and the initial condition are such that in any particular period, it is not possible to have produced more than a finite number of qualities of capital, so there is no loss of generality in this restriction. We let χ_i denote the vector consisting of one unit of quality i capital and zero units of all other qualities of capital. So, for example, $(\chi_2, 0)$ is an input vector with one unit of quality 2 capital and zero units of everything else. The period commodity space is $Z \times \Re_+$.

The set of all possible activities a is denoted by $\mathcal{A}$. An activity $a \in \mathcal{A}$ can be written as a vector consisting of a triplet $[z(a); z^+(a); c(a)]$, where $z(a), z^+(a) \in Z$, and $c(a) \in \Re_+$. Here $z(a) = [\kappa(a), \ell(a)]$ represents the input requirement for activity a in period t, $z^+(a) = [\kappa^+(a), \ell^+(a)]$ represents the output of period $t+1$ inputs produced by activity a, and $c(a)$ represents consumption produced by activity a and available in period t.

Our basic assumption is that capital of quality i can be used to produce consumption, capital of the same quality, or capital of the next higher quality. We assume that labor is an input (and also an output) in the production of consumption, but not in the production of capital. While this is just a simplifying assumption, as we discuss in the conclusion, it is consistent with the idea that there is little labor mobility between sectors. In any case, as we discuss in the conclusion, even if we allowed labor mobility, the general nature of our results would not change.

Specifically, there is a sequence of activities for producing consumption, one for each type of capital i. For quality i capital, the activity is represented by the input and output vectors $[\chi_i, 1/\gamma^i; 0, 1/\gamma^i; 1]$, $\gamma > 1$. In other words, producing a unit of consumption requires a unit of capital (of any quality) and a number of units of labor that is smaller the higher is the quality of the capital. The assumption that $\gamma > 1$ embodies the notion that technological innovation is labor saving. Notice that labor appears here both as an input and as an output, so that labor used in production today remains available for production tomorrow.

Two sequences of activities can produce capital. They are $[\chi_i, 0; \beta\chi_i, 0; 0]$, $\beta > 1$ and $[\chi_i, 0; \rho\chi_{i+1}, 0; 0]$, $\rho > 0$. This means that the current quality of capital can be used to produce either β units of the same quality of capital (capital widening) or ρ units of the next quality of capital (capital deepening). We set $\beta > \rho$, so that introducing the next quality of capital goods instead of widening the current one is costly because it requires a sacrifice of current consumption. We assume that $\mu > \min\{\beta, \gamma\}$, so maximum utility over feasible consumption paths is finite. We also assume that there is free disposal and that there is an activity that produces next-period labor by means of current-period labor $[0, 1; 0, 1; 0]$. In conjunction with the assumption that labor is an output of the activity that produces consumption, the previous assumptions guarantee that labor reproduces itself in each period; hence, labor is always available in a constant amount. We can make the labor supply grow at some rate $n < \beta - 1$ by modifying the output vectors of these two activities appropriately. Our results would not change.

The endowment $z_0 = (\kappa_0^0\chi_0, 1)$ consists of κ_0^0 units of quality 0 capital and one unit of labor.

2.1 Equilibrium

We call $\lambda \in (\times_{t=0}^{\infty}\Re_+^{\mathcal{A}})$ a *production plan* and $c \in (\times_{t=0}^{\infty}\Re_+)$ a *consumption plan.* Together they determine an (intertemporal) *allocation.*

Definition 1. *The allocation λ, c is a feasible allocation for the initial condition z_0 if for all $t \geq 0$*

$$1 \geq \sum_{a\in\mathcal{A}} \lambda_0(a)\ell(a)$$

$$\kappa_0^0\chi_0 \geq \sum_{a\in\mathcal{A}} \lambda_0(a)\kappa(a)$$

$$\sum_{a\in\mathcal{A}} \lambda_t(a)z^+(a) \geq \sum_{a\in\mathcal{A}} \lambda_{t+1}(a)z(a).$$

Definition 2. *The allocation λ^*, c^* solves the social planner problem for* *initial condition z_0 if it solves*

$$\max_{\lambda, c} \quad U(c)$$

subject to the feasibility of the allocation.

Notice that in a feasible production plan $\lambda_t(a) = 0$ if a uses as input any quality of capital greater than t. Denote by A_t the set of viable activities which use as input qualities of capital no greater than t.

Let q_t^i denote the price of quality i capital in period t, let q_t^ℓ denote the price of labor in period t, and let p_t denote the price of consumption in period t. We denote by q_t the vector of all input prices in period t, and we let q and p denote, respectively, the infinite sequence of prices of the two inputs and consumption starting in period 0. Prices q, p and a feasible allocation λ, c are a competitive equilibrium if c maximizes $U(c)$ subject to the budget constraint

$$\sum_{t=0}^{\infty} p_t c_t \leq q_0^0 \kappa_0^0 + q_0^\ell$$

and activities satisfy the zero-profit condition

$$q_{t+1} \cdot \left[\kappa^+(a), \ell^+(a)\right] + p_t c(a) - q_t \cdot [\kappa(a), \ell(a)] \leq 0, \forall a \in A_t, \ t = 0, 1, \ldots$$

with equality if $\lambda_t(a) > 0$.

In the appendix, we prove the relevant version of the first and second welfare theorems:

Theorem 1. *Suppose that λ^*, c^* is a feasible allocation for the initial condition z_0. Then λ^*, c^* solves the social planner problem if and only if we can find prices q, p such that q, p, λ^*, c^* are a competitive equilibrium.*

The following existence and uniqueness result is also proved in the appendix.

Theorem 2. *For given z_0, a competitive equilibrium exists, and there is a unique competitive equilibrium consumption plan c^*.*

We can now use the first-order conditions to give a relatively simple characterization of equilibrium consumption paths. We begin by calculating the least amount of initial capital needed to produce a given consumption in a particular period. Given a particular value of c_t, observe that either $c_t \leq 1$ or, for some $i > 0$, $\gamma^{i-1} < c_t \leq \gamma^i$. In the former case, define $\eta(c_t) = 0$; in the latter, $\eta(c_t) = i$. In this way, $\eta(c)$ indexes the highest quality of capital needed to afford a per capita consumption level equal to c. Define

$$\kappa_0^0(c_t) = \begin{cases} \beta^{-t} c_t & \eta(c_t) = 0 \\ \beta^{-t} \left(\frac{\beta}{\rho}\right)^{\eta(c_t)} \frac{\gamma c_t - \gamma^{\eta(c_t)}}{\gamma - 1} + \beta^{-t} \left(\frac{\beta}{\rho}\right)^{\eta(c_t)-1} \frac{\gamma^{\eta(c_t)} - c_t}{\gamma - 1} & \eta(c_t) > 0. \end{cases}$$

The latter expression represents the amount of initial capital required to produce c_t when it is produced using only qualities $\eta(c_t)$ and $\eta(c_t)-1$ capital. Using this expression, we also define the initial capital requirement to produce the consumption plan c

$$\kappa_0^0(c) = \sum_{t=0}^{\infty} \kappa_0^0(c_t).$$

Next we define a correspondence that captures the first-order conditions for an optimal path. Our basic strategy is to find an optimal path for a given price of initial capital q_0^0 and then back out the initial condition. First we define the constants

$$\begin{aligned} \zeta_0 &= 1 \\ \zeta_i &= (\beta/\rho)^{i-1} \left[(\beta\gamma/\rho) - 1\right] / (\gamma - 1). \end{aligned}$$

For $q_0^0 \geq 0$ and $t = 0, 1, \ldots$, we use these constants to define the correspondence $c_t' \in C_t(c_t, q_0^0)$ from $c_t \in [0, \gamma^t]$ into $\Re_+$

$$\begin{array}{ll} u'(c_t') = (\beta\delta)^{-t} q_0^0 \zeta_{\eta(c_t)} & \text{if } c_t < \gamma^{\eta(c_t)},\ \eta(c_t) \leq t \\ (\beta\delta)^{-t} q_0^0 \zeta_{\eta(c_t)} \leq u'(c_t') \leq (\beta\delta)^{-t} q_0^0 \zeta_{\eta(c_t)+1} & \text{if } c_t = \gamma^{\eta(c_t)},\ \eta(c_t) < t \\ (\beta\delta)^{-t} q_0^0 \zeta_{\eta(c_t)} \leq u'(c_t') & \text{if } c_t = \gamma^{\eta(c_t)},\ \eta(c_t) = t. \end{array}$$

This correspondence consists of horizontal and vertical line segments forming the steps of a descending stair as shown in Figure 1.

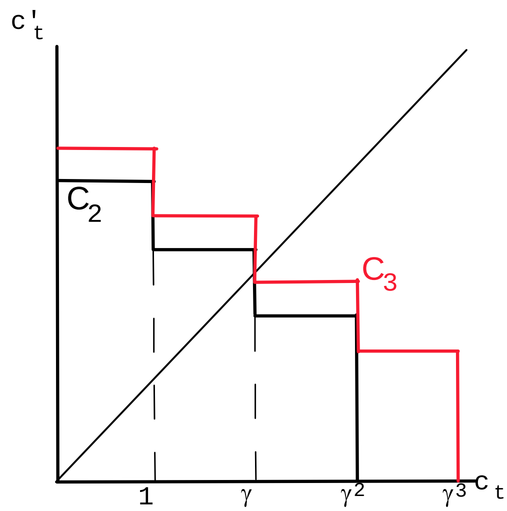

Figure 1 - Stairstep Correspondence

Because of its stairstep nature, the correspondence is upper-hemicontinuous, convex valued, and nonincreasing. It is immediate to see that, for given q_0^0 and t, it has only one fixed point, $c_t^* \in (0, \gamma^t]$. Notice that the location of the stairsteps is always at $\gamma^i, i = 0, 1, \ldots$. The stair is always truncated at γ^t since no type of capital greater than $i = t$ can be used. For this reason, as t increases, the end of the stairstep always moves to the right. Whether the vertices move up or down with t depends on $\beta\delta$. Notice that $u'(\cdot)$ is a decreasing function and $\beta\delta$ is raised to a negative power of t. So if $\beta\delta > 1$, the stairs get higher; if $\beta\delta < 1$, the stairs get lower. The figure shows C_2, C_3 in the case in which $\beta\delta > 1$. We will exploit these basic observations below in analyzing equilibrium consumption paths.

The idea behind the C correspondence is to consider that c_t determines the price of consumption through $\delta^t u'(c_t)$. Given the price of consumption and the initial price of capital q_0^0, we can calculate in period t how profitable it is to use each viable quality of capital to produce consumption. Inspection of the profit conditions shows that they are concave as a function of the index i. This means that at most two activities can yield zero profits when all others yield nonnegative profits. We then plug the prices $\delta^t u'(c_t)$, q_0^0 into the two activities that earn zero profits. Observe that using these two activities (or one activity) together with full employment yields exactly c_t' units of consumption. Consequently, the correspondence is defined by the condition that the amount of consumption giving rise to the prices should equal the amount of consumption generated from the production technology when the zero-profit conditions are satisfied. In other words, the key property of the correspondence C is that its fixed points capture the first-order conditions for an optimal path. In the appendix, we formally prove the following theorem.

Theorem 3. *For given z_0, the feasible consumption plan c^* is an optimum if and only if there exists a q_0^0 such that*

$$\kappa_0^0 \geq \kappa_0^0(c^*) \text{ with equality unless } q_0^0 = 0$$

$$c_t^* \in C_t(c_t^*, q_0^0).$$

Moreover, equilibrium prices are given by the following:

$$q_t^i = \beta^{-t} \left(\frac{\beta}{\rho}\right)^i q_0^0$$

$$w_t = \gamma^{\eta(c_t^*)} \left[\delta^t u'(c_t^*) - \beta^t (\beta/\rho)^{\eta_t} q_0^0\right]$$

and

$$q_T^\ell = \sum_{t=T}^{\infty} w_t.$$

The equilibrium production plan is any feasible plan that produces c_t^* using only capital of quality $\eta(c_t^*)$ and $\eta(c_t^*) - 1$ and has full employment whenever $\eta(c_t^*) > 0$.

3 Solow, Growth Cycles, and Stagnation

Here we focus on the long-run behavior of the economy. We show that, depending on parametric configurations and initial conditions, there are three possible long-run outcomes.

In the first, a new quality of capital is introduced every period and the economy grows at the rate γ, independently from preferences and other technological parameters. We refer to this as the Solow growth path. For the technology considered here, the Solow path provides the highest attainable level of consumption in each single period. At the opposite extreme, only the lowest quality of capital is used and the capital stock either declines or remains the same forever. We refer to this second outcome as the *stagnation* steady state. While possible in principle, both the Solow growth path and stagnation are very unlikely outcomes requiring extreme configurations of the parameter values. Finally, the economy may enter an irregular growth cycle, in which two qualities of capital are used for a period of time and then the lower quality capital is dropped and a new quality of capital is introduced for the first time, and so on. We refer to this as the *growth cycle.* This is the main focus of our interest.

We first study the Solow balanced growth path, which is the easiest, and then the growth cycle, which is the most interesting. We conclude with the special case of stagnation.

3.1 The Solow Balanced Growth Path

Improving the quality of capital does not change the amount of output that can be produced from that capital, but it does reduce the labor requirement for one unit of output. Since there is a fixed supply of labor, the economy can grow, but only by continually moving to higher qualities of capital that make it possible to produce increased amounts of output from the existing stock of labor. When the innovation occurs, ρ units of new capital are produced for each unit of old capital invested, generating an additional demand of $\rho/\gamma - 1$ units of labor input. If $\rho > \gamma$, the latter quantity is positive and, in each period, it is possible to shift the entire stock of capital from one quality to the next without causing labor to be unemployed. If, at a certain point in time, there is enough capital to employ all available labor and, from that period onward, a new quality of capital is introduced each period, the capital stock can grow fast enough that all labor remains employed on capital of the newest quality. In this case, the rate of technological innovation is independent of preferences and, also independent of preferences, consumption grows at the fixed rate γ. We refer to this as the *Solow growth path,* since this is the same result as in the Solow growth model with exogenous technological innovation.

If $\rho > \gamma$ and the initial capital stock is large enough, then the unique equilibrium is this Solow growth path beginning with consumption of a unit in period 1. Notice that if this path is feasible it must be optimal, since it is not possible to achieve higher consumption in any period by means of any other plan.

Recall that κ_0^0 is the initial stock, and let κ_t^i denote the capital stock of quality i in period t. Along a Solow growth path, at t only κ_t^t is positive. Suppose that κ_0^0 is larger or equal to the least capital stock needed for the Solow growth path to be feasible. Then we must have $\kappa_1^1 = \gamma\kappa_0^0$. In addition, a unit of capital must be used to produce one unit of consumption in period 0, so $\kappa_1^1 = \rho\left(\kappa_0^0 - 1\right)$. Solving, we find that an initial stock of capital equal, at least, to

$$\kappa_0^0 = \frac{\rho}{\rho - \gamma}$$

is needed to make the Solow path feasible. We summarize this by the following theorem.

Theorem 4. *If $\rho > \gamma$ and $\kappa_0^0 \geq \rho/(\rho - \gamma)$, the unique equilibrium is a balanced growth path in which a new technology is introduced every period, consumption in period t is γ^t, capital also grows at the rate γ, and there is full employment in all periods.*

Next we look at the behavior of prices, factor shares, and observable total factor productivity (TFP) along the Solow path. Notice first that along a Solow path, $\eta_t = t$ for all t. Further, for $c^* = \{\gamma^t\}_{t=0}^{\infty}$, the initially required capital stock is $\kappa_0^0(c^*) = \frac{\rho}{\rho-\gamma}$. Hence, we can take $q_0^0 = 0$ if $\kappa_0^0 > \frac{\rho}{\rho-\gamma}$. In fact, in this Solow case, we can normalize the initial price of capital $q_0^0 = 0$, also for $\kappa_0^0 = \frac{\rho}{\rho-\gamma}$, since utility does not increase with increases in the capital stock. This implies that the price of all qualities of capital in all periods is zero. When the marginal utility of income is normalized to $\psi = 1$, the consumption prices are

$$p_t = \delta^t u'(\gamma^t).$$

Wages are

$$w_t = \gamma^t p_t$$

and the real wage $\tilde{w}_t = w_t/p_t = \gamma^t$, so real wages grow exponentially over time.

Notice that, independently from our normalization of the initial price of capital, output grows at a constant and exogenous rate γ and factor shares are constant at the level determined in the first period. The capital/labor ratio is also growing at the constant rate γ. Similarly for effective or, as we call it here, *enhanced* labor with the productivity of physical labor growing at the exogenous rate γ. Hence, our golden age is observationally equivalent to the traditional Solow growth model, with a Cobb-Douglas production function and exogenous technological innovation.

3.2 The Growth Cycle

When circumstances are not so lucky, that is, when either $\gamma < \rho$ or the initial stock of capital $\kappa_0^0 < \rho/(\rho-\gamma)$ is too low to make the Solow path immediately accessible, both the long-run behavior of consumption and the introduction of new technologies will generally depend upon preferences and in particular on the subjective discount factor δ. There are two cases, depending on whether $\delta\beta > 1$ or $\delta\beta \leq 1$. If there were no labor constraint and no endogenous innovation, this would correspond to the case in which the equilibrium exhibits sustained growth through capital accumulation, or stagnation, with consumption eventually bounded or decreasing. As we shall see, this remains the case with a labor constraint. We take the case of a growing economy, that is, $\delta\beta > 1$, first.

3.2.1 The General Case

We begin by establishing that $\delta\beta > 1$ does correspond to sustained growth in per capita consumption and in the quality of capital. First we observe that consumption is nondecreasing:

Lemma 1. *Suppose that $\delta\beta > 1$. Then $c^*_{t+1} \geq c^*_t$.*

Proof. The correspondence C is a stairstep with vertices

$$\left(\gamma^i,\ \ [u']^{-1}\left((\beta\delta)^{-t}\, q_0^0(\beta/\rho)^{i-1}\left[(\beta\gamma/\rho)-1\right]/\left(\gamma-1\right)\right)\right).$$

Increasing $\delta\beta$ increases the height of the vertex for each γ^i. In addition, the upper bound on the domain of the correspondence, γ^t, grows larger with t as well. It follows that the fixed point c^*_t must be nondecreasing. ■

An immediate implication is that the technologies used to produce consumption must be improving over time because, with full employment, consumption would otherwise have to decrease. It is also the case that asymptotically, there is no upper bound on the quality of capital used to produce consumption.

Lemma 2. *Suppose that $\delta\beta > 1$. Then there is no upper bound on the qualities of capital used to produce consumption.*

Proof. Observe that for fixed i, as $t \to \infty$,

$$\frac{(\beta\delta)^{-t}\, q_0^0 (\beta/\rho)^{i-1} \left[(\beta\gamma/\rho) - 1\right]}{(\gamma - 1)} \to 0.$$

Hence, for any given i, for large enough t the fixed point of C must lie to the right of γ^i, meaning that a higher quality capital than i is used to produce consumption. (See Figure 1.) ■

For $\delta\beta > 1$, the general picture is therefore the following. As t grows larger, the correspondence C moves up and to the right. Observe that C has only horizontal and vertical segments. If the correspondence moves upward sufficiently slowly (which, for a given utility function, is more likely the closer $\delta\beta$ is to one), then the fixed point will generally lie on the same horizontal segment for several consecutive periods. This length of time will determine the rate at which new technologies are introduced and the speed of capital widening. The exact mechanics can be appreciated by noticing that the system behaves differently on horizontal and vertical segments. On horizontal segments, two types of capital are used to produce consumption; one of them (the highest in quality) is being accumulated while the other is being phased out. During these periods, consumption grows at a rate determined by the speed of accumulation of the highest quality capital and the correspondence C shifts upward. We refer to this as a *boom.* On vertical segments, only one type of capital is used to produce consumption, while a new quality of capital is being introduced by means of the ρ technology. Because the stock of capital used to produce consumption is not increasing, consumption remains constant as the correspondence C shifts upward with time. We refer to this period as a *growth recession.* In other words, the economy exhibits an endogenous cycle in technological innovation and, therefore, in the growth rate of TFP.

One striking fact is that during a growth recession (which corresponds to periods of technological innovation), the real wage increases together with the share of labor in national income. Specifically, during a recession, consumption is constant, so its present-value price declines by a factor δ. On the other hand, the present value price of each quality of capital declines at $1/\beta < \delta$, and in particular the real price of capital falls. Since only one activity is used to produce consumption, zero profit for this activity implies that the real wage must increase. This change in relative prices makes economic sense. During a recession, the real price of higher quality capital declines and the real wage increases, until it becomes profitable to introduce the next higher quality of capital in the production of consumption to save on labor. In this sense, technological innovation is biased in this model because it takes place to conserve a particular factor when its relative price is high enough to make the innovation profitable.

One point to emphasize is the endogeneity of technological innovation in this type of equilibrium. Although the fact that the highest quality of capital that can be used to produce consumption in period t is exogenously given, it is not this exogenous constraint that determines which technology is actually used in that period. Rather, the quality of capital i used to produce consumption in period t is generally lower than the highest exogenously available; that is, $i < t$. The exact quality of capital used in each period is determined by prices and a profit-and-loss calculation. Technologies are willingly introduced when it becomes profitable to do so for rational economic agents.

3.3 The Continuous Time Limit

We can get a more accurate picture of the cycle by studying a special case. We continue to assume that $\delta\beta > 1$ and suppose that the effective amount of time Δ between periods is small. So that the cycle does not depend on time, assume that, at least for consumption exceeding a minimum amount, preferences have the CES form

$$u(c_t) = -(1/\theta)[c_t]^{-\theta}.$$

This assumption, together with the fact that new technologies improve geometrically, gives rise, when Δ is small, to a cycle length that is essentially independent of time. Let us take $\delta = e^{-r\Delta}$, $\beta = e^{b\Delta}$, so that the assumption $\delta\beta > 1$ corresponds to $b > r$. We also assume that innovations are discrete, so that the extent to which machine i saves on labor relative to machine $i-1$ is independent of the time between periods. Hence $\gamma > 1$ independently of Δ. We also have $\rho = \tilde{\rho}e^{d\Delta}$. Since innovations are costly, we assume that

$b > d$ and that $\tilde{\rho} < 1$. Because we are interested only in small values of Δ, we can also assume that $\rho = \tilde{\rho} e^{d\Delta} < \gamma$ for Δ in the range considered here. We denote calendar time by $\tau = t\Delta$.

Suppose that at some particular time $c_t^* = \gamma^{i-1}$. Then

$$u'(c_t^*) = (\beta\delta)^{-t} \, q_0^0 (\beta/\rho)^{i-1} \left[(\beta\gamma/\rho) - 1\right] / \left(\gamma - 1\right).$$

This corresponds to the beginning of a horizontal segment or a boom. In our special case, we can write this as

$$c_t^* = \left\{ \left(e^{(b-r)\Delta}\right)^{-\tau/\Delta} q_0^0 \left[\left(\frac{e^{(b-d)\Delta}}{\tilde{\rho}}\right)^{i-1} \frac{\gamma e^{(b-d)\Delta} - \tilde{\rho}}{\tilde{\rho}(\gamma - 1)} \right] \right\}^{-1/(\theta+1)} .$$

As τ increases, so does c_t^* until eventually $c_t^* = \gamma^i$, at which point the recession occurs. We can calculate a good approximation to this length of time by taking the continuous time limit when $\Delta \to 0$

$$c_t^* = \left[e^{(r-b)\tau} q_0^0 (1/\tilde{\rho})^{i-1} \frac{(\gamma/\tilde{\rho}) - 1}{\gamma - 1} \right]^{-1/(\theta+1)} .$$

In other words, during the growth period consumption is simply growing at the rate $(b - r)/(\theta + 1)$. The length of the boom τ_b is determined by the amount of time required for consumption to grow by a factor of γ, or

$$\tau_b = \frac{\theta + 1}{b - r} \ln \gamma.$$

The recession, on the other hand, lasts from t to $t + \tau_r/\Delta$, where

$$(\beta\delta)^{-t} \, q_0^0 (\beta/\rho)^{i-1} \left[(\beta\gamma/\rho) - 1\right] / \left(\gamma - 1\right)$$

$$= (\beta\delta)^{(t+\tau_r/\Delta)} \, q_0^0 (\beta/\rho)^{i} \left[(\beta\gamma/\rho) - 1\right] / \left(\gamma - 1\right).$$

The continuous time approximation gives

$$e^{(r-b)\tau_r} \frac{e^{(b-d)\Delta}}{\tilde{\rho}} = 1.$$

Taking the limit for $\Delta \to 0$ and solving for τ_r, we have that

$$\tau_r = -\frac{\ln \widetilde{\rho}}{b-r}.$$

Consider next the length $\tau_c = \tau_b + \tau_r$ of the whole cycle. This is

$$\tau_c = \frac{1}{b-r}\left[\ln(\frac{\gamma^{1+\theta}}{\tilde{\rho}})\right]$$

which is increasing in γ, θ, and r and decreasing in b and $\tilde{\rho}$. The shorter the cycle, the more quickly new technologies are introduced, so we find that the frequency of innovations responds negatively to the quality of the innovation γ, the preference parameter θ, and the subjective degree of impatience r. The most interesting of these is the quality of the innovation γ. Higher quality innovations in this model lead to less innovation, because they make it possible to grow for a longer period of time without hitting the labor constraint. On the other hand, we find that there is more innovation if the cost of producing capital, as measured by the inverse of either b or $\tilde{\rho}$, goes down.

The relative length of the two phases, booms and recessions, is

$$\frac{\tau_b}{\tau_r} = -(1+\theta)\frac{\ln \gamma}{\ln \tilde{\rho}}.$$

Interestingly enough, neither the productivity of the capital widening technology nor the degree of impatience affects the relative length of booms and recessions. Economies where people exhibit low willingness to substitute consumption over time (high values of θ) have longer (but less rampant) booms for a given recession length. As we noted above, improved quality of innovation (high γ) makes it possible to grow for a longer period of time without hitting the labor constraint. This increases the length of booms, but not of recessions. Finally, a large cost of innovation is bound to increase the relative amount of time spent in recession.

The average growth rate of consumption over an entire cycle is the value of g that solves

$$\gamma = \exp\left[g \cdot \frac{1}{b-r}\ln\left(\frac{\gamma^{1+\theta}}{\tilde{\rho}}\right)\right]$$

which is

$$g = \frac{b - r}{1 + \theta - \ln \tilde{\rho} / \ln \gamma}.$$

Hence, economies where people are more willing to substitute consumption over time grow faster on average, as do economies able to implement more substantial innovations.

We already noted above that the real wage grows during a recession. In fact, we have somehow argued that growth recessions in our model are brought about by the increase in the real wage relative to the price of new capital. To save on expensive labor by introducing the relatively cheaper new machines, resources must temporarily be shifted to the labor saving innovation, which reduces the growth rate of consumption. Because we have constructed our model in such a way that there is always full employment, this implies a countercyclical movement in the labor share of national income. Over the entire cycle, productivity of labor grows by a factor of γ, the same for the real wage. Because consumption is constant during recessions, its price relative to both old and new capital must be increasing then. Overall, the price of a machine of quality i decreases over time relative to that of consumption, and the rate of decrease is uniform across qualities.

3.4 Stagnation

Finally, we turn to the case in which $\delta\beta \leq 1$. In the absence of a labor constraint, this would imply that the economy remains stagnant, either with constant consumption if $\delta\beta = 1$ or with declining consumption if $\delta\beta < 1$. With the labor constraint, if $\rho > \gamma$ and $\kappa_0^0 \geq \rho/(\rho - \gamma)$, we have already indicated that the equilibrium is the Solow path of exogenous sustained growth regardless of whether $\delta\beta \leq 1$. In this case, introducing a labor constraint and the possibility of factor saving technological innovation, changes a stagnant economy in which consumption never grows into an expanding economy in which consumption grows forever and new technologies are introduced every period. When labor saving innovations are feasible, the addition of a labor constraint can be seen as the incentive toward adopting technological innovations that lead to higher consumption.

On the other hand, the next theorem shows that if either the Solow path does not exist because $\rho < \gamma$ or there is insufficient initial capital, then the picture is indeed one of a stagnant economy in the long run. There is an upper limit on the highest quality of capital ever produced, and ultimately consumption either stops growing ($\delta\beta = 1$) or declines ($\delta\beta < 1$).

Theorem 5. *Suppose that either $\rho < \gamma$ or $\kappa_0^0 < \rho/(\rho - \gamma)$. If $\delta\beta \leq 1$, there exists a technology I such that no quality of capital greater than I is ever produced and a period T such that for all $t \geq T$, the following are true.*

- *If $\delta\beta = 1$, $c_t^* = c_T^*$.*
- *If $\delta\beta < 1$, $c_{t+1}^* < c_t^* < 1$.*
- *Only the lowest quality capital is used to produce consumption.*

Proof. Under the conditions given, it follows that $q_0^0 > 0$. As $i \to \infty$, the Inada condition for $c \to \infty$ implies that

$$\begin{aligned}&[u']^{-1}\left((\delta\beta)^{-t} q_0^0 (\beta/\rho)^{i-1}\left[(\beta\gamma/\rho) - 1\right]/(\gamma - 1)\right)\\ &\leq [u']^{-1}\left(q_0^0 (\beta/\rho)^{i-1}\left[(\beta\gamma/\rho) - 1\right]/(\gamma - 1)\right) \to 0.\end{aligned}$$

It follows that there is some technology I for which, for all t,

$$[u']^{-1}\left((\delta\beta)^{-t} q_0^0 (\beta/\rho)^{I-1}\left[(\beta\gamma/\rho) - 1\right]/(\gamma - 1)\right) < \gamma^I.$$

Consequently, no technology $i \geq I$ is used to produce consumption. It follows that the optimal consumption plan does not ever produce any capital of quality $i \geq I$.

For $\delta\beta = 1$, the correspondence C does not increase or decrease, it simply shifts to the right; once $t \geq I$, it follows that there is a unique and time-independent fixed point of C. For $\delta\beta < 1$, as $t \to \infty$ we have $[u']^{-1}\left((\delta\beta)^{-t} q_0^0\right) \to 0$, so eventually the fixed point of C must lie below one. Since $[u']^{-1}\left((\delta\beta)^{-t} q_0^0\right)$ is also strictly decreasing, so is c_t^*. Since the fixed point of C is going to zero, it must eventually fall below one, with the implication that only the lowest possible quality technology is used to produce consumption. ■

This theorem demonstrates another important possibility in this economy: path-dependence. That is, suppose that $\delta\beta < 1$ and $\rho > \gamma$. Then if initial capital exceeds the level $\kappa_0^0 \geq \rho/(\rho - \gamma)$ needed for the Solow path, the long-run possibility is one of technological innovation and sustained growth. On the other hand, if initial capital falls a bit short of the threshold, so that $\kappa_0^0 < \rho/(\rho-\gamma)$ in the long run, only the lowest possible quality capital is used, there is unemployment, and consumption continually falls. In particular, if we compare two economies with different initial capital endowments, one above and one below the threshold, we discover that they do not converge to the same long-run growth path.

Finally, we point out a further interesting property of our economy: consumption and growth paths that, depending upon initial conditions, may be strictly nonmonotone. More precisely, even when $\kappa_0^0 < \rho/(\rho-\gamma)$, the economy may innovate and grow for some period of time, before falling back into stagnation. For economies of this kind, relatively rich at the beginning but highly impatient or not very efficient at reproducing already existing capital, consumption will grow at a rate $\gamma > 1$ for a while and then decline forever. Because the decline is governed by the stairstep correspondence C, the decline is uneven until only the lowest possible quality of capital is employed, at which time (in the CES case) consumption declines geometrically. It is as if the airplane gets off the runway, then falls back to the ground. Figure 2 illustrates three cases, associated with different initial stocks of capital and $\rho > \gamma$. The black monotone upward sloping curve describes the consumption path associated to the Solow case. The combination of the first portion of the Solow consumption path with either one of the downward sloping curves represents different cases of transient growth followed by decline and eventual stagnation. Obviously, in the latter two cases, the initial amount of capital has to be lower than $\rho/(\rho-\gamma)$ because, when $\rho > \gamma$, if the initial amount of capital is sufficiently high, the economy reaches the Solow growth path and remains there forever.

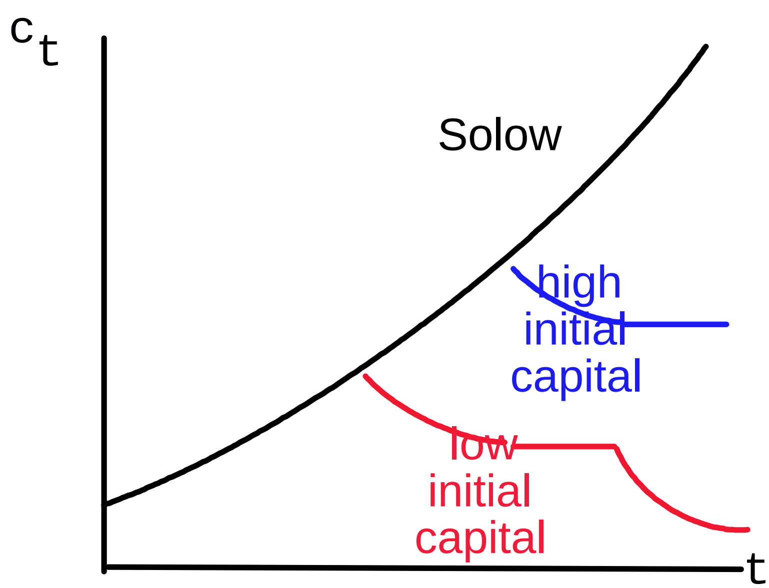

Figure 2 - Stagnation Consumption Paths

4 Conclusion

We have examined a model in which an essential input cannot be increased at the same speed as other inputs. Hence, growth in per capita consumption can take place if and only if factor saving innovations are possible. An innovation, such as a new machine, is factor saving when it reduces the input requirement of some factor per unit of output. Machines which, for given output levels, need fewer inputs than other machines must be more expensive. Hence, factor saving innovations necessarily induce a nontrivial trade-off between capital widening and capital deepening. Capital widening is less costly, but eventually hits a factor constraint (labor in our example), forcing growth through capital deepening. Consequently, the rate at which new technologies are introduced becomes endogenous, depending on, among other things, the rate of intertemporal substitution in consumption, the technology, and the economy's initial conditions. As in other models of endogenous growth, the rate of growth of consumption is also endogenous in the same sense.

At least since Hicks' (1932) seminal writings, economists have been debating about factor saving innovations that are biased. The term indicates that technological innovation augments productivity for some factor or factors more than for others and that it does so because of relative price incentives. We have built a general equilibrium model capturing this intuition and looked at its implications. Our main finding is that, in these circumstances, technological innovation is likely to be endogenous and, indeed, affected by relative prices and initial conditions. Further, we have proved that when technological innovation is factor saving, it *must* come in cycles unless very special circumstances occur.

We have chosen to model the factor constraint as binding in the consumption sector only and to concentrate on the case of just one fixed factor. However, the basic message remains the same regardless of such simplifying restrictions. Obviously, when the scarce factor can grow at a rate n, the whole analysis can be replicated for $\beta > 1 + n$. When more than one scarce factor exists, factor saving innovations can take place along different directions. While this can complicate the model and its equilibrium dynamics, providing an interesting topic for future extensions, the basic message about the oscillatory nature of factor saving technological innovation would only be strengthened. Finally, the basic message does not depend on the sector in which the constraint binds or on whether labor is perfectly mobile between the two sectors. Indeed, we worked out preliminary versions of the case of perfect labor mobility between the two sectors, without qualitatively different results. Notice, incidentally, that our example does not require that there be no labor used in the production of capital, just that there is labor immobility between the two sectors and that the labor constraint binds first in the consumption sector. This particular example is a useful starting point because of its simplicity and the stark results it delivers. In addition, we do not think the assumption of perfect labor mobility between the two sectors is especially more plausible than complete immobility between the two sectors.

Appendix

Here we prove Theorems 1, 2, and 3 from the preceding paper. We start with the following lemma.

Lemma A. *A consumption plan c, with $c_t > 0$ for all t, maximizes $U(c)$*

subject to the budget constraint if and only if for some $\psi \geq 0$,

$$p_t = \psi \delta^t u'(c_t)$$
$$\sum_{t=0}^{\infty} p_t c_t = q_0^0 \kappa_0^0 + q_0^\ell.$$

Proof. This is standard. ■

Proof of Theorem 1. That a competitive equilibrium solves the social planner problem is a standard first welfare theorem proof. To prove the second, we need to show that we can find prices that support a solution to the social planner problem.

Suppose that λ, c^* is a solution to the planner problem for the initial condition z_0. Let z^* be the corresponding inputs. Let z_{T+1} denote a vector of labor and capital of quality $i \leq T+1$. Let $V^{T+1}(z_{T+1})$ denote the maximum utility, discounted at $t = 0$, of beginning with the endowment z_{T+1} in period $T+1$ and continuing forward. Let $U^T(c) = \sum_{t=0}^{T} \delta^t u(c_t)$. Observe that λ^*, c^* solves the problems of maximizing $U^T(c) + V^{T+1}(z_{T+1})$ subject to social feasibility. This is a finite-dimensional problem. By standard finite-dimensional arguments, we can find finite-dimensional price vectors $\overline{q}^T, \overline{p}^T$ so that the zero-profit conditions are satisfied for viable activities up to $T+1$. By the same standard arguments, the vector $c_t^*, t = 0, \ldots, T$ and the scalar z_{T+1}^* are optimal for the consumer under the budget constraint $\sum_{t=0}^{T} \overline{p}_t^T c_t + \overline{q}_{T+1}^T z_{T+1} = \overline{q}_0^{\kappa T} \kappa_0^0 + \overline{q}_0^{\ell T}$.

Now we can normalize prices so that $\overline{p}_t = \delta^t u'(c_t^*)$. Let Q^T denote the set of all (nonnegative) infinite-dimensional price sequences for which the projection on $\Re_+^T \times \Re_+^{2\times(T+1)}$ is a supporting price vector for the finite-dimensional problem above. Observe that $Q^T \supseteq Q^{T+1}$ and that these are closed spaces. It follows that $Q = \cap_{T\to\infty} Q^T$ is closed, although possibly empty.

Next observe that $\overline{q}_{t+1}^T$ is a supergradient of $V^{t+1}(z_{t+1})$ at z_{t+1}^*. Notice that $\overline{q}_{t+1}^T$ is bounded below by zero and above by some finite two-dimensional vector as $\overline{q}_{t+1}^T z_{t+1}^* + V^{t+1}(0) \leq V^{t+1}(z_{t+1}^*)$ and that the latter is finite. It follows that the intersection Q is nonempty.

Let q be in Q. By construction q and p (which is uniquely defined from the first-order condition) satisfy the zero-profits condition. From the consumer budget constraint in the truncated problems, we have $\sum_{t=0}^{T} p_t c_t^* + q_{T+1} z_{T+1}^* = q_0^\kappa \kappa_0^0 + q_0^\ell$. Since $\overline{q}_{t+1}$ is a gradient of $V^{t+1}(z_{t+1})$ at z_{t+1}^*, we have $q_{T+1} z_{T+1}^* + V^{T+1}(0) \leq V^{T+1}(z_{t+1}^*)$. Also $V^{T+1}(0) = \delta^{T+1} u(0)/(1-\delta)$

and $V^{T+1}(z^*_{T+1}) = \sum_{t=T+2}^{\infty} \delta^t u(c^*_t)$. Since $\sum_{t=0}^{\infty} \delta^t u(c^*_t) < \infty$, it follows that $\lim_{T\to\infty} V^{T+1}(z^*_{T+1}) \to 0$ and so $q_{T+1}z^*_{T+1} \to 0$, which gives $\sum_{t=0}^{\infty} p_t c^*_t = q^0_0\kappa^0_0 + q^\ell_0$. ■

Proof of Theorem 2. Since $U(c)$ is bounded above on the feasible set of feasible consumption paths, it is continuous in the product topology. Since this set is compact in the product topology, an optimum exists; it is unique since U is strictly concave. ■

Define a *simple plan* to be a pair of sequences of integers $(\nu, \eta) = (\nu_0, \eta_{0,}\nu_1, \eta_1, \dots)$, where $\nu_t \in \{1, 2\}$, $\nu_0 = 1$; $t \geq \eta_t \geq 0$; and $\eta_t > 0$ if $\nu_t = 2$. A production plan $(\lambda,\ k)$ is *consistent* with the simple plan (ν, η) if the following are true.

1. Exactly ν_t different qualities of capital are employed in period t to produce consumption.

2. When $\nu_t = 1$, the quality of capital employed to produce consumption is η_t.

3. When $\nu_t = 2$, the two qualities of capital used to produce consumption are $\eta_t, \eta_t - 1$.

We say that a production plan exhibits *full employment* if there is unemployment only in periods where no quality of capital other than zero is used to produce consumption. We say that a simple plan (ν, η) and a consumption stream c are *consistent* if there is a full-employment production plan (λ, k) consistent with the simple plan that yields the output c. If $\nu_t = 1$ and $\eta_t = 0$, then (λ, k) uses exactly $\kappa^{\eta_t}_t = c_t \leq 1$ units of quality 0 capital. If $\nu_t = 1$ and $\eta_t > 0$, then the plan uses $\kappa^{\eta_t}_t = \gamma^{\eta_t}$ units of quality η_t capital for full employment, so $c_t = \gamma^{\eta_t}$. If $\nu_t = 2$, then the plan uses exactly $\kappa^{\eta_t}_t$ units of quality η_t capital and exactly $\kappa^{\eta_t-1}_t$ units of quality $\eta_t - 1$ capital, where these are the unique solutions of $\kappa^{\eta_t}_t/\gamma^{\eta_t} + \kappa^{\eta_t-1}_t/\gamma^{\eta_t-1} = 1$ and $\kappa^{\eta_t}_t + \kappa^{\eta_t-1}_t = c_t$. For convenience, we now replicate the definitions from the text. For any given value of c_t, observe that either $c_t \leq 1$ or for some $i > 0$, $\gamma^{i-1} < c_t \leq \gamma^i$. In the former case, define $\eta(c_t) = 0$; in the latter, $\eta(c_t) = i$. Let

$$\kappa^0_0(c_t) = \begin{cases} \beta^{-t} c_t & \eta(c_t) = 0 \\ \beta^{-t}\left(\frac{\beta}{\rho}\right)^{\eta(c_t)} \frac{\gamma c_t - \gamma^{\eta(c_t)}}{\gamma - 1} + \beta^{-t}\left(\frac{\beta}{\rho}\right)^{\eta(c_t)-1} \frac{\gamma^{\eta(c_t)} - c_t}{\gamma - 1} & \eta(c_t) > 0 \end{cases}$$

and

$$\kappa_0^0(c) = \sum_{t=0}^{\infty} \kappa_0^0(c_t).$$

Set

$$\begin{aligned}\zeta_0 &= 1\\ \zeta_i &= (\beta/\rho)^{i-1}\left[(\beta\gamma/\rho) - 1\right] / (\gamma - 1)\,.\end{aligned}$$

Define the correspondence $c'_t \in C_t(c_t, q_0^0)$ by

$$\begin{array}{ll} u'(c'_t) = (\beta\delta)^{-t}\, q_0^0 \zeta_{\eta(c_t)} & \text{if } c_t < \gamma^{\eta(c_t)},\ \eta(c_t) \le t \\ (\beta\delta)^{-t}\, q_0^0 \zeta_{\eta(c_t)} \le u'(c'_t) \le (\beta\delta)^{-t}\, q_0^0 \zeta_{\eta(c_t)+1} & \text{if } c_t = \gamma^{\eta(c_t)},\ \eta(c_t) < t \\ (\beta\delta)^{-t}\, q_0^0 \zeta_{\eta(c_t)} \le u'(c'_t) & \text{if } \ c_t = \gamma^{\eta(c_t)},\ \eta(c_t) = t. \end{array}$$

This correspondence is upper-hemicontinuous, convex valued, and nonincreasing.

Proof of Theorem 3. For given z_0, suppose that the feasible consumption plan c^* is an optimum. We first claim that there is an initial price of capital q_0^0, a nonnegative sequence of wages $w = (w_0, w_1, \dots)$, and a simple plan (ν, η) consistent with c^* such that the following conditions hold.

(1) $\delta^t u'(c_t^*) - \beta^{-t}(\beta/\rho)^{\eta_t} q_0^0 - w_t/\gamma^{\eta_t} = 0$.

(2) If $\nu_t = 2$, then $\delta^t u'(c_t^*) - \beta^{-t}(\beta/\rho)^{\eta_t - 1} q_0^0 - w_t/\gamma^{\eta_t - 1} = 0$.

(3) If $\nu_t = 1$ and $\eta_t < t$, then $\delta^t u'(c_t^*) - \beta^{-t}(\beta/\rho)^{\eta_t + 1} q_0^0 - w_t/\gamma^{\eta_t + 1} \le 0$.

(4) If $\nu_t = 1$ and $\eta_t > 0$, then $\delta^t u'(c_t^*) - \beta^{-t}(\beta/\rho)^{\eta_t - 1} q_0^0 - w_t/\gamma^{\eta_t - 1} \le 0$.

(5) $w_t = 0$ if there is unemployment at t.

First observe that if capital of quality $i > 0$ is used to produce consumption for period t and there is unemployment, strictly greater consumption in that period can be had by replacing quality i capital in period t with quality 0 capital. The full-employment condition is consequently necessary for an optimum along any path that uses capital of quality other than $i = 0$.

We now apply the zero-profit conditions for competitive equilibrium. Let q_t^i denote the price of quality i capital in period t, and let q_t^ℓ denote the price of labor. If we begin with one unit of quality 0 capital in period 0, then we need i quality upgrades producing ρ units of capital each and $t - i$ periods producing β units of capital to get from quality 0 in period 0 to quality i in period t, for all possible $i \le t$. The order in which the ρ and β phases come does not matter. It follows then, from the zero-profit condition applied to the capital-producing activities only, that if $i \leq t$, then

$$q_t^i = \frac{q_0^0}{\rho^i \beta^{t-i}} = \beta^{-t} \left(\frac{\beta}{\rho}\right)^i q_0^0.$$

From the fact that labor can always reproduce itself, we have

$$q_t^\ell \geq q_{t+1}^\ell$$

with equality if there is unemployment in period t. So we can define the wage rate as

$$w_t = q_t^\ell - q_{t+1}^\ell \geq 0.$$

We can then write the profits from the activity that in period t produces consumption c_{t+1} from quality i capital as

$$\pi_t^i = \delta^t u'(c_t) - \beta^{-t}(\beta/\rho)^i q_0^0 - w_t/\gamma^i.$$

Recall that in equilibrium profits must be nonpositive. Observe that this function is strictly concave as a function of i for fixed values of c_t, q_0^0, and w_t. It follows that if this function is nonpositive for all $i \leq t$, it is zero for at most two activities, in which case it is strictly negative for all other activities. If it is zero for one activity, it is sufficient that it be nonpositive for the next highest and next lowest activities to be nonpositive for all activities. So since in equilibrium $p_t = \delta^t u'(c_t^*)$ conditions (1)-(5) are indeed necessary.

Next we observe that $c_t^* \in C_t(c_t^*, q_0^0)$ if and only if (1)-(4) hold. In the case $c_t < \gamma^{\eta(c_t)}$, full employment requires that c_t be produced using qualities $\gamma^{\eta(c_t)}, \gamma^{\eta(c_t)-1}$ of capital. Then $v_t = 2$, and (1) and (2) must hold. Solving yields $u'(c_t^*) = (\beta\delta)^{-t} q_0^0 \zeta_{\eta(c_t^*)}$ and

$$w_t = \frac{\beta^{-t}\left[(\beta/\rho)^{\eta(c_t^*)} - (\beta/\rho)^{\eta(c_t^*)-1}\right] q_0^0}{1/\gamma^{\eta(c_t^*)-1} - 1/\gamma^{\eta(c_t^*)}}$$

which is nonnegative since $\beta/\rho \geq 1$ and $1/\gamma < 1$.

Turning to $c_t = \gamma^{\eta(c_t)}$, we have (1), (3), and (4):

$$\begin{aligned}
\delta^t u'(c_t^*) - \beta^{-t}(\beta/\rho)^{\eta(c_t^*)} q_0^0 - w_t/\gamma^{\eta(c_t^*)} &= 0 \\
\delta^t u'(c_t^*) - \beta^{-t}(\beta/\rho)^{\eta(c_t^*)+1} q_0^0 - w_t/\gamma^{\eta(c_t^*)+1} &\leq 0 \\
\delta^t u'(c_t^*) - \beta^{-t}(\beta/\rho)^{\eta(c_t^*)-1} q_0^0 - w_t/\gamma^{\eta(c_t^*)-1} &\leq 0.
\end{aligned}$$

We can solve the first equation for w_t. Substituting into the second inequality, we see that it is satisfied if and only if $u'(c^*) \leq (\beta\delta)^{-t} q_0^0 \zeta_{\eta(c_t)}$, and the first if and only if $(\beta\delta)^{-t} q_0^0 \zeta_{\eta(c_t)-1} \leq u'(c^*)$. It is easy to check that these two inequalities also imply that $w_t \geq 0$.

Finally, observe that $\kappa_0^0 \geq \kappa_0^0(c^*)$ with equality unless $q_0^0 = 0$ since otherwise it would be possible to produce c^* with less than the initial capital stock.

This proves that the conditions of Theorem 3 are necessary for an equilibrium. To show that they are also sufficient, observe that the Inada conditions imply that $u'(\cdot)$ maps $[0, \infty)$ onto itself; hence, for every $q_0^0 > 0$, there is a κ_0^0 for which q_0^0 is the equilibrium price of capital. Let c^* be the corresponding optimal consumption. This satisfies the necessary conditions $c_t^* \in C_t(c_t^*, q_0^0)$ and $\kappa_0^0 = \kappa_0^0(c^*)$. Since $c_t^* \in C_t(c_t^*, q_0^0)$ has a unique solution, it follows that these conditions are sufficient as well. ■

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