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SUSTAINABLE PLANS AND MUTUAL DEFAULT

V. V. Chari*   Patrick J. Kehoe*

Federal Reserve Bank of Minneapolis
and University of Minnesota

ABSTRACT

This paper presents a simple general equilibrium model of optimal taxation in which both private agents and the government can default on their debt. As a benchmark we consider *Ramsey* equilibria in which the government can precommit to its policies at the beginning of time, but in which private agents can default. We then consider *sustainable* equilibria in which both government and private agent decision rules are required to be sequentially rational. We adapt Abreu's (1988) optimal trigger strategies to completely characterize the entire set of sustainable equilibria. In particular, we show that when there is sufficiently little discounting and government consumption fluctuates enough, the Ramsey allocations and policies (in which the government never defaults) can be supported by a sustainable equilibrium. By way of an example we show that the larger the variance of government consumption the easier it is to support good outcomes. We also show for moderate discount factors that in the best sustainable equilibrium tax rates are more volatile and debt is smaller than in the Ramsey equilibrium.

A classic problem in the time consistency literature is that of government default on debt. In an early contribution, Prescott (1977) analyzed a simple infinite horizon economy in which the government finances a given stream of expenditures by raising distorting labor taxes and uses debt to smooth tax distortions over time. He found that if there is no technology for committing to its actions, the government will always default on outstanding debt to reduce distorting taxes. In the equilibrium of his model, the value of government debt is zero and the government runs a continuously balanced budget. This early work presents a challenge to economists interested in explaining why governments do not default on their inherited debt. The conventional explanation is that governments fear that if they default, private agents will be less willing to lend to them in the future. If the losses incurred by the government from future borrowing difficulties outweigh the current benefits from defaulting, the government will not default.

In this paper and a companion paper (Chari and Kehoe 1989), we explore this conventional explanation in the context of a formal general equilibrium model. Our main insight is that whether or not this explanation can account for government debt depends crucially on the set of enforcement technologies available. In our earlier paper, we followed the standard setup of the time consistency literature in assuming that the government had no way of committing to honor its debt, but private agents were forced to honor their debts. Rather surprisingly, in that model the conventional explanation does not hold. In the model, there are equilibria in which, if the government ever defaulted on its debt, private agents would never let the government borrow from them again. By assumption, however, the government could always safely lend to private agents. The ability to lend undercuts the force of the borrowing restrictions. It turns out that, even if the government could never borrow after it defaults, it could smooth tax distortions by lending to private agents. Hence, the losses incurred from banning the government from future borrowing are not sufficient to deter the government from defaulting on its debt.

In this paper we consider a model in which both private agents and the government can default on their debt. The idea is that while it is easy to imagine environments in which the government can, at least partially, default on its debt—say, by taxation or inflation—it is also easy to imagine that certain kinds of debt owed to the government by private citizens, such as student loans or loans to farmers, are hard to collect. To keep the analysis tractable, we consider the extreme situation in which the government cannot enforce any

private debt claims but can perfectly enforce tax payments. We find that here, in stark contrast to our earlier work, the conventional explanation does hold. First, there are equilibria in which, if the government defaults, private agents never let it borrow from them again. Moreover, since it cannot enforce debt payments by private agents, the government cannot safely lend to the private sector. Thus, there are equilibria in which, if the government ever defaults, it is stuck with continuously balancing its budget thereafter. If it is forced to balance its budget at each date, the government cannot smooth tax distortions at all, and this leads to a rather low level of welfare. Thus, here the losses incurred from banning the government from future borrowing are large. Under plausible assumptions, we show that these losses are large enough to deter the government from ever defaulting.

Formally, our model is a variant of the optimal fiscal policy models of Prescott (1977), Barro (1979), Lucas and Stokey (1983), and Persson, Persson, and Svensson (1987). We begin by characterizing the equilibrium under commitment, called the *Ramsey equilibrium* as a solution to a planning problem. We then consider an environment without commitment in which we allow the allocation rules of consumers and policy plans of the government to depend on the whole history of past government policies. We define a *sustainable equilibrium* as a set of allocation rules and policy plans that satisfy sequential rationality conditions for both the private agents and the government. We adapt some techniques from game theory, especially Abreu's (1988) optimal trigger strategies, to characterize the entire set of sustainable allocations and policies. We use this characterization to show that if the fluctuations in government consumption never damp out and the discount rate is sufficiently small, then even the Ramsey outcomes can be supported by a sustainable equilibrium. We go on to show how the supportability of Ramsey outcomes depends critically on the nature of fluctuations in government spending. We provide examples showing that, if the fluctuations eventually damp out, then the Ramsey outcomes cannot be supported, regardless of the discount factor. Notice that our results are quite different from the standard results in repeated games. This difference arises because our model is a dynamic game with debt as a state variable. In the model the desire by the government to smooth tax distortions over different dates with different levels of government spending drives all the results. Notice, in particular, that if our model was repeated, in the sense that government consumption was constant, then there would be no role for debt in the first place.

This paper is related to the literature on international default (for example, Eaton, Gersovitz, and Stiglitz 1986; Grossman and Van Huyck 1986; Manuelli 1986; Cole and English 1988; Bulow and Rogoff 1989; and Atkeson 1991). This literature investigates whether a threat to cut off a defaulting country from future borrowing is credible and, if so, whether it can support positive borrowing by that country. Bulow and Rogoff find that such threats cannot support positive borrowing, whereas Grossman and Van Huyck find that they can. The key to understanding these contrary results is the set of enforcement technologies available. Bulow and Rogoff assume there is a technology through which agents in one of the countries, the home country, can commit to servicing its debt, while there is no such technology in the foreign country. In this setup, no matter what it has done, the foreign country can always safely lend to the home country's agents to smooth consumption. Bulow and Rogoff use this feature to prove that no equilibrium can have positive borrowing by the foreign country. In contrast, Grossman and Van Huyck explicitly rule out the possibility of the foreign country ever lending to the home country. In their equilibrium, if the foreign country defaults, it is forced into autarky for some (stochastic) length of time. Grossman and Van Huyck show that these threats are credible and that they can support borrowing by the foreign country. Thus, the link between enforcement technologies and the ability to support positive debt shows up in the open economy literature also.

1. The Economy

Consider a simple production economy populated by a large number of identical infinitely lived consumers. In each period t, there are two goods: labor and a consumption good. A constant returns-to-scale technology is available to transform one unit of labor into one unit of output. The output can be used for private consumption or for government consumption. The per capita level of government consumption in each period, denoted g_t, is exogenously specified. Let c_t and ℓ_t denote the per capita levels of private consumption and labor. Feasibility requires that

(1.1) $c_t + g_t = \ell_t.$

The preferences of each consumer are given by

(1.2) $\sum_{t=0}^{\infty} \beta^t U(c_t, \ell_t)$

where $0 < \beta < 1$ and U is increasing in consumption, decreasing in labor, and strictly concave and bounded. Assume also that the utility function satisfies the Inada conditions.

Government consumption is financed through a proportional tax on labor income and with debt. Let τ_t denote the tax rate on labor income in period t. A unit of government debt at t is a claim to one unit of the consumption good at $t + 1$. Let b_t denote the number of claims purchased by consumers and q_t denote the price of a claim. Let $\delta_t \in [0,1]$ denote the default rate on government debt outstanding in period t. Here $\delta_t = 0$ corresponds to complete repayment, $\delta_t = 1$ to complete default, and $0 < \delta_t < 1$ to partial default. (One can think of δ_t as a tax on debt.) The consumer's budget constraints can then be written as

$$(1.3) \qquad c_t - (1-\tau_t)\ell_t + q_t b_t = (1-\delta_t)b_{t-1} \quad \text{for } t = 0, ..., \infty$$

with $b_{-1} = 0$. An allocation is a sequence $x = (x_t)_{t=0}^{\infty}$, where $x_t = (c_t, \ell_t, b_t)$.

The government sets labor tax rates, default rates, and debt prices to finance an exogenous sequence of government consumption. The preferences of the government are given by (1.2). The government's budget constraint is

$$(1.4) \qquad \tau_t \ell_t - g_t + q_t b_t = (1-\delta_t)b_{t-1} \quad \text{for } t = 0, ..., \infty.$$

A policy is a sequence $\pi = (\pi_t)_{t=0}^{\infty}$, where $\pi_t = (\tau_t, \delta_t, q_t)$.

2. Commitment

Consider an environment with the following institutional structure. First, suppose there is an institution or commitment technology through which the government can bind itself to a policy once and for all at time zero. In particular, the government can commit to never defaulting on its debt. The commitment technology is formalized by having the government choose a policy once and for all and then having consumers choose their allocations. Next suppose that claims purchased by consumers are nonnegative so

$$(2.1) \qquad b_t \geq 0 \quad \text{for all t.}$$

This assumption allows the government to borrow from consumers but not to lend to them. One motivation for this assumption is that private agents are anonymous so that debt claims against them are unenforceable. Clearly, in such an environment, if an individual is lent a positive amount, the individual will default; thus, the current price of a claim to an individual's future payment is zero. The anonymity of private agents,

however, need not preclude the government from raising tax revenues. For example, the tax in our economy can be interpreted as a payroll tax which is levied directly on firms who pay the anonymous agents after tax wages. Such a tax does not require that individual agents are identifiable.

Here, a policy is an infinite sequence of numbers $\pi = (\pi_t)_{t=0}^{\infty}$. Since the government needs to predict how consumers will respond to its policies, consumer behavior is described by rules that associate government policies with allocations. An allocation rule is a sequence of functions $f = (f_t)_{t=0}^{\infty}$ that maps policies into allocations. A *Ramsey equilibrium* is a policy π and an allocation rule f that satisfy the following conditions: (i) For every policy π', the allocation $f(\pi')$ maximizes (1.2) subject to (1.3) and (2.1); (ii) the policy π maximizes $\sum_{t=0}^{\infty}\beta^t U(c_t(\pi),\ell_t(\pi))$ subject to $\tau_t\ell_t(\pi) - g_t + q_t b_t(\pi) = (1-\delta_t)b_{t-1}(\pi)$ for all t.

The allocations in a Ramsey equilibrium solve a simple programming problem called the *Ramsey problem*. We let R_t denote the value of the government surplus at t; namely, $R_t = U_c(\tau_t\ell_t - g_t)$. In any equilibrium, we can use the consumer's first-order conditions together with feasibility to write this as $R_t = U_c c_t + U_\ell \ell_t$. We have, then,

<u>Proposition 1</u> (The Ramsey Equilibrium). The consumption and labor allocations, c and ℓ, in the Ramsey equilibrium solve the Ramsey problem which is to maximize $\sum_{t=0}^{\infty}\beta^t U(c_t,\ell_t)$ subject to (1.1),

(2.2) $$\sum_{t=0}^{\infty}\beta^t R_t = 0$$

(2.3) $$\sum_{t=s}^{\infty}\beta^t R_t \geq 0 \quad \text{for } s = 1, 2, \ldots.$$

<u>Proof</u>. In the Ramsey equilibrium the government must satisfy its budget constraints taking as given the allocation rule f. These requirements impose restrictions on the set of allocations the government can obtain by varying its policies. We claim that the restrictions are summarized by (1.1), (2.2), and (2.3). We first show that these restrictions imply (1.1), (2.2), and (2.3). First, note that by adding (1.3) and (1.4) we get (1.1), and thus feasibility is satisfied in equilibrium. Next, consider the allocation rule f. For any policy π', using the Inada conditions and Weitzman's Theorem (1973) the necessary and sufficient conditions for c, ℓ, and b to solve the consumer's problem are the consumer's budget constraint and

(2.4) $\beta^t U_c(c_t,\ell_t) = p_t$,

(2.5) $\beta^t U_\ell(c_t,\ell_t) = -p_t(1-\tau_t)$,

(2.6) $p_t q_t \geq (1 - \delta_{t+1}) p_{t+1}$,   with equality if $b_t > 0$

(2.7) $\lim_{t \to \infty} p_t q_t b_t = 0$

where, for each t, p_t is the Lagrange multiplier on constraint (1.3). Multiplying (1.3) by p_t, summing over

t, and using (2.6) and (2.7) gives

(2.8) $\displaystyle\sum_{s=t}^{\infty} p_s [c_s - (1 - \tau_s)\ell_s] = p_t(1 - \delta_t)b_{t-1}.$

Using (2.4) and (2.5), we can write (2.8) as

(2.9) $\displaystyle\sum_{s=t}^{\infty} \beta^s [U_c c_s + U_\ell \ell_s] = \beta^t U_c (1 - \delta_t)b_{t-1}$

where, for convenience, we have suppressed the arguments of the derivatives. Since $b_t \geq 0$, the right side

of (2.9) is nonnegative. Therefore, in any equilibrium, (2.3) must hold. Since $b_{-1} = 0$ it follows that (2.2)

must hold. Thus, (1.1), (2.2), and (2.3) are implied by the requirements that the government must satisfy its

budget constraint and that allocations are consistent with the allocation rule f.

Next, given any set of allocations c and ℓ that satisfy (1.1), (2.2), and (2.3) we can construct policies

such that these allocations are consistent with the allocation rule f and the government's budget constraint.

The consumer's first order conditions imply the tax rate on labor income is given by

(2.10) $1 - \tau_t = -U_\ell(c_t, \ell_t)/U_c(c_t, \ell_t).$

The debt prices, debt, and default rates are not uniquely determined, as inspection of (2.6) and (2.8) make

clear. One way to construct them is to set the default rate identically to zero and to set the debt prices by

(2.11) $q_t = \beta^t U_c(c_{t+1}, \ell_{t+1})/U_c(c_t, \ell_t).$

The debt is then given by

(2.12) $\beta^t U_c(c_t, \ell_t)b_{t-1} = \displaystyle\sum_{s=t}^{\infty} \beta^s R_s.$

Thus (1.1), (2.2), and (2.3) characterize the restrictions imposed by the government's budget constraint and

the allocation rule f. Since in the Ramsey equilibrium the government chooses a policy that maximizes the

welfare of consumers it follows that the Ramsey allocations solve the Ramsey problem. $\square$

For later it will be useful to say a policy-allocation pair (π, x) is *attainable under commitment* if it satisfies consumer optimality and the government budget constraint. In the proof of the proposition we have shown that a policy-allocation pair is attainable under commitment if and only if the allocations satisfy (1.1), (2.2), and (2.3). The associated policies can then be constructed as in the proposition. Next, note that if the inherited debt at date zero given by b_{-1} is positive, the government will default and the Ramsey problem is unchanged. Since we require that the debt be nonnegative the assumption that $b_{-1} = 0$ is without loss of generality. Finally, as equation (2.12) makes clear the left side of (2.3) is the value of the inherited debt at date s.

3. No Commitment

Consider an environment in which no commitment technologies are available to the government. Formally, the government's lack of commitment is modeled by having the government choose policy sequentially as opposed to once and for all at date 0. In each period, the government and the consumers can vary their decisions, depending on the history of government policies up to the time the decision is made. At the beginning of period t, the government chooses a current policy as a function of the history $h_{t-1} = (\pi_s | s = 0, \ldots, t-1)$ together with a contingency plan for setting future policies for all possible future histories. Let $\sigma_t(h_{t-1})$ denote the time t labor tax rate, default rate, and price of debt chosen by the government when faced with history h_{t-1}. After the government sets current policy, consumers make their decisions. Faced with a history $h_t = (h_{t-1}, \pi_t)$, consumers choose time t levels of consumption, labor supply, and debt holdings, denoted $f_t(h_t)$, together with a contingency plan for choosing future allocations for all possible future histories. (The reader may wonder why the histories do not include consumers' decisions. For a discussion of this point see Chari and Kehoe 1990.)

To define a sustainable equilibrium, we need to explain how policy plans induce future histories. For any policy plan $\sigma = (\sigma_0, \sigma_1, \ldots)$, let $\sigma^t = (\sigma_t, \sigma_{t+1}, \ldots)$ denote the sequence of policy rules from time t onwards. For any t, call σ^t the continuation of σ. Let f^t denote the corresponding objects for the allocation rules. Given a history h_{t-1}, the policy plan σ induces future histories by $h_t = (h_{t-1}, \sigma_t(h_{t-1}))$ and so on.

Consider the situation of the government in period t. Given some history h_{t-1} and given that future allocations evolve according to f, the government chooses a policy plan σ^t that maximizes the welfare of consumers $\sum_{s=t}^{\infty} \beta^s U(c_s(h_s), \ell_s(h_s))$ subject to

(3.1) $\qquad \tau_s(h_{s-1})\ell_s(h_s) - g_s + q_s(h_{s-1})b_s(h_s) = (1 - \delta_s(h_{s-1}))b_{s-1}(h_{s-1})$

where for all $s \geq t$ the future histories are induced by σ from h_{t-1}.

Consider a private agent in period t. Given some history h_t and given that future policies evolve according to σ, a consumer chooses an allocation rule f^t to maximize $\sum_{s=t}^{\infty} \beta^s U(c_s(h_s), \ell_s(h_s))$ subject to

(3.2) $\qquad c_t(h_t) - (1-\tau_t)\ell_t(h_t) + q_t b_t(h_t) = (1-\delta_t)b_{t-1}(h_{t-1})$

and, for $s > t$,

(3.3) $\qquad c_s(h_s) - (1 - \tau_s(h_{s-1}))\ell_s(h_s) + q_s(h_{s-1})b_s(h_s) = (1 - \delta_s(h_{s-1}))b_{s-1}(h_{s-1})$

where π_t is given in h_t and for all $s > t$ the future histories are induced by σ from h_t.

A *sustainable equilibrium* is a pair (σ, f) that satisfies the following conditions: (i) Given a policy plan σ, the continuation of the allocation rule f solves the consumer's problem for every history h_t; (ii) given the allocation rule f, the continuation of policy plan σ solves the government's problem for every history h_{t-1}. Notice that in the definition, we require that both the consumers and the government act optimally for every history of policies—even for histories not induced by the government's policy plan. This requirement is analogous to the requirement of perfection in a game.

We turn to a characterization of the set of sustainable outcomes. We start by considering a finite horizon model. We use backward induction to show that in any sustainable equilibrium the government's budget constraint is continuously balanced and, thus, tax distortions cannot be smoothed over time. In the last period, if the government inherits positive debt, it will default on it to minimize the amount of revenues it must raise through the distorting labor tax. Anticipating these policies, consumers in the next to last period will not buy any government debt at a positive price and the government will default on inherited debt and choose taxes to balance its budget. The same argument holds for earlier periods. We call the associated policy plans and allocation rules the *autarky plans* and specify them as follows. For any history, the government's policy plan specifies $\delta_t^a = 1$ and $q_t^a = 0$. The labor tax rate, denoted $\tau^a(g_t)$, is given by this solution to this

problem: maximize $U(c,\ell)$ subject to $c + g_t = \ell$ and $(1+U_\ell/U_c)\ell = g_t$ where $1 - \tau^a(g_t) = -U_\ell/U_c$. Let $U^a(g_t)$ denote the maximized value of this problem and call it the *autarky utility level*. The allocation rules for consumers specify that, for every history, debt holdings are zero and consumption and labor supply maximize utility given the current tax rate.

In the infinite horizon model, the way to characterize the set of equilibria is not obvious. One way to proceed is to take the limit of a sequence of finite horizon equilibria. This technique will indeed yield an equilibrium. There are many other equilibria, however, that are not the limit of any sequence of finite horizon equilibria. In fact, the set of sustainable equilibria is very large and difficult to characterize. Fortunately, it is relatively easy to characterize the policies and allocations induced by such equilibria. Recall that a sustainable equilibrium (σ, f) is a sequence of functions that specify policies and allocations for all possible histories. Starting from date zero, a sustainable equilibrium induces a particular sequence of polices and allocations, say, (π, x). We call this the *outcome* induced by the sustainable equilibrium.

In the next proposition, we characterize the entire set of sustainable outcomes. In characterizing the set of such outcomes, we extend Abreu's (1988) seminal work on optimal punishments in repeated games to our dynamic framework. To prove the proposition, we use a set of plans called the *revert-to-autarky* plans which will turn out to be the analogues of Abreu's optimal punishments. For an arbitrary sequence of policies (π, x), the revert-to-autarky plans specify continuation with the candidate sequences (π, x) as long as the specified policies have been chosen in the past; otherwise, they specify revert to the autarky plans (σ^a, f^a). Thus, for example, at time t given a history h_{t-1}, this policy plan specifies this: Choose the policies π_t specified by π if the policies $(\pi_0, \ldots, \pi_{t-1})$ have been chosen according to π. If they have not, then revert to the autarky policy plan σ^a. The revert-to-autarky allocation rules are similarly defined. We then have

Proposition 2 (Sustainable Equilibrium Outcomes). An arbitrary pair of sequences (π, x) is an outcome of a sustainable equilibrium if and only if (i) the pair (π, x) is attainable under commitment, and (ii) for every t, the following inequality holds:

$$(3.4) \qquad \sum_{s=t}^{\infty} \beta^s U(c_s, \ell_s) \geq \sum_{s=t}^{\infty} \beta^s U^a(g_s).$$

Proof. Suppose, first, that (π,x) is the outcome of a sustainable equilibrium (σ,f). Consumer optimality requires that x maximize consumer welfare at date zero, while government optimality implies that π satisfies the government's budget constraint. Thus, (π,x) is attainable under commitment. Next, in any sustainable equilibrium, the consumers, when confronted with the autarky policies, choose consumption and labor allocations to solve a simple static problem. More precisely, for any history of the form $h_t = (h_{t-1}, \pi_t^a)$, the allocations $c_t(h_t)$ and $\ell_t(h_t)$ solve the problem of maximizing $U(c,\ell)$ subject to $c \leq (1-\tau)\ell$. From this result, it follows that for any history h_{t-1} a deviation by the government from σ to σ^a for all periods from t onwards is feasible in that σ^a satisfies the government budget constraint for *any* equilibrium allocation rule f. Clearly, then, the utility of the government must be at least as large as the right side of (3.4) for every period t. Thus (i) and (ii) hold.

Now suppose that some arbitrary pair of sequences (π,x) satisfies (i) and (ii). We show that the associated revert-to-autarky plans constitute a sustainable equilibrium. Consider, first, histories under which there have been no deviations from π up until t. Since x is optimal for consumers at date zero when they are faced with π, the continuation of x is optimal for consumers at date t when they are faced with the continuation of π. Consider the situation of the government. If it deviates at date t then consumers will revert to the autarky allocation rules from time t onward. By construction, when the government is faced with the autarky allocation rule, it is optimal for it to choose the autarky policies. Hence, the most a deviation by the government at t can attain is the right side of (3.4). If the assumed inequality holds, then sticking to the specified plan is optimal. Consider, next, histories in which there have been deviations from π before t. The revert-to-autarky plans then specify that both the government and the consumers pursue the autarky plans forever. Clearly such plans are sustainable. $\square$

It is important to emphasize that it is a gross misreading of Proposition 2 to think that it only characterizes the set of outcomes sustainable by a particular trigger strategy. On the contrary, it characterizes the *entire set* of outcomes that can be supported by *any* conceivable sustainable equilibrium. One can think of the revert-to-autarky plans as simply a convenient mathematical device used in constructing this set. Our proposition implies that in checking whether a candidate outcome is sustainable by *any* sustainable plan, we need only check that this outcome, which is simply a sequence of numbers, satisfies the conditions of the

proposition rather than searching over the complicated space of plans, each of which is a sequence of history contingent functions. We exploit this feature in proving that, under certain conditions, the Ramsey outcomes are sustainable. Notice that in the proposition, in the spirit of Abreu's work, we have in fact established that the autarky equilibrium is the worst equilibrium and that an allocation is sustainable by *any* equilibrium if and only if it is sustainable by triggering to this worst equilibrium after deviations.

An immediate corollary of Proposition 2 is the following:

<u>Corollary</u> (The Optimal Sustainable Outcome). The sustainable outcome with the highest date 0 utility solves the problem: choose $\{c_t, \ell_t\}$ to maximize (1.2) subject to (1.1), (2.2), (2.3), and (3.4).

Thus the optimal sustainable outcome solves a programming problem similar to the Ramsey problem except that the optimal sustainable outcome must also satisfy the dynamic incentive constraints (3.4) which capture the restrictions that sequential rationality places on equilibrium outcomes.

4. Uncertainty

Here we extend the analysis of the previous sections to allow for stochastic government consumption. We first set up the model with uncertainty and show how the results of the previous sections extend to it. Then we establish some characteristics of the Ramsey outcome under the assumption that government consumption follows a positive Markov process. Finally, we use these characteristics to show that the Ramsey plan is sustainable for large discount factors.

Consider the model with stochastic government consumption. Let government consumption follow a given stochastic process for which the realizations up to and including time t are $g^t = (g_0, \ldots, g_t)$. The probability of observing any particular state g^t is $\mu(g^t)$. The initial realization g_0 is given. Each realization g_t is assumed to be in a finite set $\{\gamma_1, \ldots, \gamma_m\}$ with $\gamma_1 < \ldots < \gamma_m$. There is no other uncertainty in the economy, so the natural space of allocations is the space of infinite sequences $(c, \ell, b) = \{c(g^t), \ell(g^t), b(g^t) \mid \text{for all t and } g^t\}$, where $c(g^t)$, $\ell(g^t)$, and $b(g^t)$ are contingent on the state g^t. We define similar objects for the policy π. An allocation is feasible if

$$(4.1) \qquad c(g^t) + g_t = \ell(g^t) \quad \text{for all t and } g^t.$$

The preferences of consumers are given by the expected utility function

(4.2) $\displaystyle\sum_t \sum_{g^t} \beta^t \mu(g^t) U(c(g^t), \ell(g^t)).$

The consumer's budget constraint at time t in state g^t is

(4.3) $\quad c(g^t) - (1 - \tau(g^t))\ell(g^t) + \displaystyle\sum_{g_{t+1}} q(g^t, g_{t+1}) b(g^t, g_{t+1}) = (1 - \delta(g^t))b(g^t).$

The government's budget constraint is similarly defined. Notice that, as in Lucas and Stokey (1983), the government issues state-contingent debt. At each date t and state g^t the government sells a bundle of claims, one for each realization of g at t + 1. Each claim $b(g^t, g_{t+1})$ pays off only if g_{t+1} is realized. This type of debt allows the government to insure itself against fluctuations in government consumption.

A Ramsey equilibrium is defined analogously to that in Section 2. From the arguments in Proposition 1, it is clear that the Ramsey equilibrium allocations maximize (4.2) subject to the constraints (4.1),

(4.4) $\displaystyle\sum_{t=0} \sum_{g^t} \beta^t \mu(g^t) R(g^t) = 0, \quad$ and

(4.5) $\displaystyle\sum_{t=s}^{\infty} \sum_{g^t \ni g^s} \beta^t \mu(g^t) R(g^t) \ge 0 \quad$ for all s and g^s

where $R(g^t) = U_c c(g^t) + U_\ell \ell(g^t)$ and $g^t \ni g^s$ means all those states g^t whose first s + 1 components are g^s. Next, we incorporate history into the environment without commitment by letting the histories include the past realizations of government spending. For such an environment, it is immediate to prove the analogue of Proposition 3: an arbitrary pair of contingent sequences (π, x) is an outcome of a sustainable equilibrium if and only if the pair is attainable under commitment and for each s and g^s the following inequality holds:

(4.6) $\displaystyle\sum_{t=s}^{\infty} \sum_{g^t \ni g^s} \beta^t \mu(g^t) U(c(g^t), \ell(g^t)) \ge \sum_{t=s}^{\infty} \sum_{g^t \ni g^s} \beta^t \mu(g^t) U^a(g_t).$

The obvious corollary is that the optimal sustainable allocations maximize (4.2) subject to the constraints on the Ramsey allocations (4.1), (4.4), (4.5) together with the sustainability constraint (4.6).

Thus, the Ramsey outcomes are sustainable if and only if, after every possible sequence of realizations of government consumption, the discounted value of utility from then onward under the Ramsey plan is higher than it is under the autarky plan. We call these truncated discounted values of utilities the *tail*

utilities of the Ramsey plan and the autarky plan. The autarky plan consists of a collection of static plans for which each utility level depends only on the current value of government spending. Hence, it is relatively easy to evaluate its tail utilities. In contrast, the Ramsey plan is inherently dynamic, and it is more difficult to say much about its tail utilities. Indeed, the hard part of what follows is establishing that the Ramsey plan has enough stationarity so that comparisons between it and the autarky plan can be made. Under the assumption that government consumption follows a positive Markov chain we show that the Ramsey outcome exhibits a form of stationarity which we exploit in showing that the Ramsey outcome is sustainable. Consider,

Assumption 1 (A Positive Markov Chain). Government consumption follows a Markov chain with strictly positive elements. Let $\mu > 0$ denote the minimum transition probability.

As noted before our Ramsey allocation problem is similar to the one in Lucas and Stokey (1983) except that we have additional nonnegativity constraints on debt given in (4.5). Consider for a moment the Ramsey problem omitting these constraints. The first order conditions for this problem are then the same as those in Lucas and Stokey, namely,

$$(4.7) \quad U_c(c(g^t),c(g^t) + g_t) + U_\ell(c(g^t),c(g^t) + g_t)$$

$$+ \lambda_0[R_c(c(g^t),c(g^t) + g_t) + R_\ell(c(g^t),c(g^t) + g_t)] = 0$$

where λ_0 is the Lagrange multiplier on (4.4). Notice that this equation is of the form $F(c,g_t,\lambda_0) = 0$ and thus it defines the optimal allocation $c(g^t)$ as some time invariant function of the current realization g_t and the Lagrange multiplier λ_0. Using (4.1) it follows that $\ell(g^t)$ is also a time invariant function of g_t and λ_0 and thus, so is the optimal surplus $R(g^t)$ which we write as $\bar{R}(g_t,\lambda_0)$. Next note that the value of the debt sold from period s to s + 1 is given by the left-side of (4.5). Under the assumption that government consumption is Markov the value of the debt under the Ramsey plan can be written as

$$(4.8) \quad \bar{R}(g_s,\lambda_0) + \beta \sum_{v=0}^{\infty} \sum_{g^v} \beta^v \mu(g^v | g_s)\bar{R}(g_v,\lambda_0).$$

Clearly the optimal debt depends only on the current value of g. Recall that, by assumption, the inherited debt at date 0 is 0. Letting γ_k denote the initial state g_0, the debt inherited in the state γ_k after any string g^t is thus the same as the inherited debt at date 0, namely zero. Thus, for all g^t the debt $b(g^t,\gamma_k) = 0$.

Note the insurance feature of the state-contingent debt emphasized by Lucas and Stokey. Even if the economy experiences a long sequence of high realizations of government consumption the debt does not build up. In particular, the debt inherited in γ_k is zero for *any* previous history g^t. Specifically, suppose government consumption takes on two values a high one in wartime and a low one in peacetime. Suppose the economy starts off in wartime. It is easy to show that under the Ramsey policies the state-contingent government debt has no value if the war continues and has a positive value if there is peace in the next period. In any peacetime the government sells debt which has no value if there is war next period and has a positive value if there is peace. (See Lucas and Stokey's Examples 7, 8, and 9 for more on the insurance role of state-contingent debt.)

It turns out that for our problem, with the nonnegativity constraints, government debt exhibits a weaker form of stationarity. Specifically, along any string along which the nonnegativity constraint has never bound the inherited debt in state γ_k is zero, that is $b(g^t,\gamma_k) = 0$ where γ_k is the initial state. Moreover, at any state where the nonnegativity constraint binds the Ramsey problem restarts with that state acting as a new initial state with zero inherited debt. We prove this stationarity result in several steps. We begin with a simple lemma about the structure of the optimal surplus function under the Ramsey plan. This surplus function is defined by the first-order conditions to the Ramsey problem. Let λ_0 denote the Lagrange multiplier on the government's budget constraint (4.4) and $\lambda_s(g^s)$ the Lagrange multiplier on the nonnegativity constraint (4.5). Then we can write these first-order conditions as (4.1), (4.4), (4.5), and

(4.9) $U_c(c(g^t),\ell(g^t)) + U_\ell(c(g^t),\ell(g^t)) + [\lambda_0 + \alpha(g^t)][R_c(g^t) + R_\ell(g^t)] = 0$

where $\alpha(g^t) = \sum_{s=1}^{\infty}[\lambda_s(g^s) | g^t \ni g^s]$ and $R_c(g^t)$ and $R_\ell(g^t)$ denote the partial derivatives of $R(g^t)$.

It is worth pointing out an implication of this first order condition which we will use several of our proofs. Suppose that $\alpha(g^t) = 0$, so that along the string g^t the nonnegativity constraint on debt has never bound, then (4.9) reduces to (4.7) and thus the Ramsey allocations at g_t depend solely on the last element of the string, namely g_t (along with the multiplier λ_0). Thus, if g^t and g^s are two strings of possibly different lengths with both $\alpha(g^t)$ and $\alpha(g^s)$ equal to zero and $g_t = g_s$ then the Ramsey allocations at g^t and g^s are identical and $R(g^t) = R(g^s)$. Notice that this feature holds in our economy along such strings for exactly the same reason it holds in the Lucas and Stokey economy.

In the lemma, we establish the following property of the surplus function $R(g^t)$ which we will use to prove Proposition 3. Consider two strings of government consumption that end in the same value. If the Lagrange multipliers have never bound on the first string but have bound on the second, then the surplus under the first string must be smaller than the surplus under the second. Formally, we have

<u>Lemma 1</u>. If, for any two strings g^t and g^s with $g_t = g_s$, the solution to the Ramsey problem has $\alpha(g^t) = 0$ and $\alpha(g^s) > 0$, then it also has $R(g^t) < R(g^s)$.

<u>Proof</u>. If (c,ℓ) solves the Ramsey problem with $\alpha(g^t) = 0$ and $\alpha(g^s) > 0$, then, with other allocations fixed, $(c(g^t),\ell(g^t))$ and $(c(g^s),\ell(g^s))$ must solve

$$(4.10) \qquad \max \beta^t \mu(g^t) U(c(g^t),\ell(g^t)) + \beta^s \mu(g^s) U(c(g^s),\ell(g^s))$$

subject to

$$(4.11) \qquad \beta^t \mu(g^t) R(g^t) + \beta^s \mu(g^s) R(g^s) + K_0 = 0$$

$$(4.12) \qquad \beta^s \mu(g^s) R(g^s) + K(g^r) \geq 0 \quad \text{for each } g^r \in g^s, r = 1, ..., s$$

and (4.1) evaluated at g^s and g^t. Here K_0 is the sum of the discounted value of surpluses at all nodes except for g^t and g^s, and for each $g^r \in g^s$, $K(g^r)$ is the sum of the discounted value of surpluses at all nodes following g^r except for g^s. Notice that (4.11) is simply (4.4), and (4.12) represents those constraints in (4.5) in which g^s appears. Notice also that, since $\alpha(g^t) = 0$, we can exclude the nonnegativity constraints in which g^t appears. Since $\alpha(g^s) > 0$, however, at least one of the nonnegativity constraints in (4.12) must bind.

Consider for a moment a less constrained problem than (4.10), with the nonnegativity constraints in (4.12) dropped. From the resulting first-order conditions, it follows that the optimal allocations at g^t and g^s denoted by $c^*(g^t)$, $\ell^*(g^t)$, and $c^*(g^s)$, $\ell^*(g^s)$ are identical, and thus the optimal surpluses—say, $R^*(g^t)$ and $R^*(g^s)$—satisfy $R^*(g^t) = R^*(g^s)$. From (4.11) we have

$$(4.13) \qquad [\beta^t \mu(g^t) + \beta^s \mu(g^s)] R^*(g^t) = -K_0 = \beta^t \mu(g^t) R(g^t) + \beta^s \mu(g^s) R(g^s).$$

Now suppose $R(g^t) \geq R(g^s)$. Then, from (4.13) we have $R(g^t) \geq R^*(g^t) = R^*(g^s) \geq R(g^s)$. But then the surplus $R^*(g^s)$ satisfies (4.12) and the allocations $(c^*(g^t),\ell^*(g^t)) = (c^*(g^s),\ell^*(g^s))$ are also feasible for problem (4.10). Since at least one of the nonnegativity constraints in (4.12) binds, we have a contradiction. $\square$

The intuition for this lemma is straightforward. If there were no nonnegativity constraints on debt the optimal policy would smooth tax distortions by setting these surpluses equal at any two states in which the levels of government consumption are equal. Now if the nonnegativity constraints bind in one of these states, say g^s, and not in the other, say g^t, it must be that the optimal surplus is bigger in the state in which it binds. If this were not the case then the original surpluses could not have been optimal since it would be possible to sell a little more debt into g^s and a little less debt into g^t, meet the nonnegativity constraints on debt and move the values of $R(g^t)$ and $R(g^s)$ closer together and in the process raise utility.

In the next proposition we use this lemma to show that if government consumption follows a positive Markov chain, then the Ramsey plan exhibits a weak form of stationarity: after any sequence of realizations there is some following state into which zero debt is sold. In the proof we use the fact that at some arbitrary g^t the value of the debt, namely $U_c(g^t)b(g^t)$, is given by the analogue of (2.12),

$$(4.14) \qquad R(g^t) + \beta \sum_{v=0}^{\infty} \sum_{g^v} [\beta^v \mu(g^t,g^v | g^t)R(g^t,g^v)].$$

We will also use the notation that for any g^t

$$(4.15) \qquad A(g^t) = \{g^v | \alpha(g^t,g^v)=0, v=0,1,...\}$$

where (g^t,g^v) denotes a $t + 1$ element string g^t starting from g_0 followed by a $v + 1$ element string g^v which may not start from g_0. In words, $A(g^t)$ is the set of strings which emanate from g^t and for which the nonnegativity constraint never binds. (Of course, if $\alpha(g^t) > 0$, $A(g^t)$ is empty.) Since zero debt is sold from states in $A(g^t)$ to states outside of $A(g^t)$, the sum over the present value of surpluses over all nodes (g^t,g_v) with $g_v \notin A(g^t)$ is zero. Thus the value of the debt at g^t is given by,

$$(4.16) \qquad U_c(g^t)b(g^t) = R(g^t) + \beta \sum_{v=0}^{\infty} \sum_{g^v} [\beta^v \mu(g^t,g^v | g^t)R(g^t,g^v) | g^v \in A(g^t)].$$

Proposition 3 (A Characteristic of Ramsey Outcomes). For every g^t there is some γ, possibly depending on g^t, such that $b(g^t,\gamma) = 0$.

Proof. For any string g^t, either the nonnegativity constraints have never bound for any $g^s \in g^t$, implying that $\alpha(g^t) = 0$, or a constraint has bound for at least one $g^s \in g^t$, implying that $\alpha(g^t) > 0$. Consider, first, a string g^t with $\alpha(g^t) = 0$. We eventually show that $b(g^t,\gamma_k) = 0$, where γ_k denotes g_0. To

show this, we first establish: If $\alpha(g^t) = 0$ then $\Delta(\gamma_k) = \Delta(g^t,\gamma_k)$. In words, if the nonnegativity constraints do not bind anywhere along the string g^t then the binding pattern of the nonnegativity constraints following (g^t,γ_k) is identical to the binding pattern after the initial state γ_k.

It turns out that establishing this preliminary result is the hard part of the proof. Consider, first, proving that if $\alpha(g^t) = 0$ then $\Delta(\gamma_k) \subset \Delta(g^t,\gamma_k)$. By way of contradiction, suppose that $\alpha(g^t) = 0$ and $\alpha(g^r) = 0$ from some r, but $\alpha(g^t,g^r) > 0$. Consider the first point along the string (g^t,g^r) such that the nonnegativity constraint binds, say, (g^t,g^s). If the nonnegativity constraint binds, then $b(g^s) = 0$. We will use Lemma 2 to argue that the value of debt sold into state g^s is strictly positive, thus contradicting the hypothesis that the nonnegativity constraint binds at this point. Using the Markov assumption the value of the debt at (g^t,g^s) is

$$(4.17) \qquad R(g^t,g^s) + \beta \sum_{v=0}^{\infty} \sum_{g^v} \beta^v \mu(g^s,g^v \mid g^s) R(g^t,g^s,g^v).$$

We claim that (4.17) is strictly greater than value of the debt at g^s namely

$$(4.18) \qquad R(g^s) + \beta \sum_{v=0}^{\infty} \sum_{g^v} [\beta^v \mu(g^s,g^v \mid g^s) R(g^s,g^v) \mid g^v \in \Delta(g^s)]$$

which by construction is nonnegative. Consider (4.17). Since the debt sold into any state is nonnegative, the present value of surpluses starting at nodes (g^t,g^s,g^v) with $g^v \notin \Delta(g^s)$ is nonnegative. Thus (4.17) is greater than or equal to

$$(4.19) \qquad R(g^t,g^s) + \beta \sum_{v=0}^{\infty} \sum_{g^v} [\beta^v \mu(g^s,g^v \mid g^s) R(g^t,g^s,g^v) \mid g^v \in \Delta(g^s)].$$

By Lemma 2, $R(g^t,g^s) > R(g^s)$ and for all $g^v \in \Delta(g^s)$, $R(g^t,g^s,g^v) > R(g^s,g^v)$ and thus (4.19) is strictly greater than (4.18). Thus, the debt at (g^t,g^s) is strictly positive, which contradicts the hypothesis. The proof of reverse inclusion is nearly identical.

We use this result to prove that $b(g^t,\gamma_k) = 0$. We claim that $b(g^t,\gamma_k)$ equals the value of the inherited debt at time zero, $b(\gamma_k)$, which by assumption is zero. Consider expressions similar to (4.16) for $b(\gamma_k)$ and $b(g^t,\gamma_k)$. First, from the above result, $\Delta(\gamma_k) = \Delta(g^t,\gamma_k)$, so both sums are over the same values of g^v. Second, since the constraints do not bind along all such strings, using the argument following (4.9), it follows that the surpluses at (γ_k,g^v) and (g^t,γ_k,g^v) are equal. Third, from the Markov assumption, the probability weights in the two sums are identical, thus $b(g^t,\gamma_k) = b(\gamma_k) = 0$.

Consider, next, a string with $\alpha(g^t) > 0$. Consider the last point in the string at which the nonnegativity constraint bound, say, at $g^s \in g^t$ with $g_s = \gamma_\ell$. The problem from this point on is exactly the same as the original problem with γ_ℓ replacing γ_k. We can, then, use exactly the same argument as before, treating γ_ℓ as the initial node to conclude that $b(g^t, \gamma_\ell) = 0$. $\square$

We next investigate the sustainability of the Ramsey outcomes under an additional assumption which guarantees, basically, that some tax-smoothing is optimal for large discount factors. Consider, then,

Assumption 2 (Tax-Smoothing). There are two states γ_i and γ_j such that $R(\gamma_i) < 0 < R(\gamma_j)$, where $R(\gamma_i)$ and $R(\gamma_j)$ are given in the solution to the following problem: choose (c_i, ℓ_i) and (c_j, ℓ_j) to maximize $U(c_i, \ell_i) + \mu_{ij} U(c_j, \ell_j)$ subject to

$$c_k + \gamma_k = \ell_k \quad \text{for } k = i, j$$

$$R(\gamma_i) + \mu_{ij} R(\gamma_j) = 0$$

$$R(\gamma_j) \geq 0$$

where μ_{ij} denotes the probability of going from state i to state j.

We will show that, under Assumptions 1 and 2, the Ramsey outcomes are sustainable when there is sufficiently little discounting. We begin with a lemma which states that the difference in utility under the Ramsey and autarky plans becomes arbitrarily large as the discount factor approaches one. We let $V^r(\gamma, \beta)$ and $V^a(\gamma, \beta)$ denote the discounted utility under the Ramsey and autarky plans, respectively, starting from an initial state γ with a discount factor β. We have

Lemma 2. Under Assumptions 1 and 2, for every constant M, there is a $\underline{\beta}$ in $(0,1)$ such that, for all β in $(\underline{\beta}, 1)$ and all initial states γ, $V^r(\gamma, \beta) - V^a(\gamma, \beta) \geq M$.

Proof. Let the initial state g_0 be some $\gamma \in \{\gamma_1, \ldots, \gamma_k\}$. Let γ_i and γ_j be the states that satisfy Assumption 2. Let $c_k(\beta)$ and $\ell_k(\beta)$ for $k = i, j$ denote the optimal allocations and $R_k(\beta)$ for $k = i, j$ denote the surpluses for the problem in Assumption 2 with μ_{ij} replaced by $\beta\mu_{ij}$. Since these optimal surpluses are continuous functions of β, Assumption 2 guarantees there is a neighborhood of one—say, $(\underline{\beta}, 1)$—such that, for all $\beta \in (\underline{\beta}, 1)$, $R_j(\beta) > 0$. Thus, there is some $\epsilon > 0$ such that, for all $\beta \in (\underline{\beta}, 1)$

(4.20) $\quad [U(c_i(\beta),\ell_i(\beta)) + \beta\mu_{ij}U(c_j(\beta),\ell_j(\beta))] - [U(c_i^a,\ell_i^a) + \beta\mu_{ij}U(c_j^a,\ell_j^a)] > \epsilon$

where (c_k^a,ℓ_k^a), $k = i, j$, denote the autarky solutions.

Consider the following plan. For strings g^{t+1} with $g_t = \gamma_i$ and $g_{t+1} = \gamma_j$, let the allocations at g_t and g_{t+1} be given by $(c_i(\beta),\ell_i(\beta))$ and $(c_j(\beta),\ell_j(\beta))$. For all other strings g^{t+1}, let the allocations at g_t and g_{t+1} be the autarky allocations. Denote the present value of the plan by $V(\gamma,\beta)$. It is clear that this plan does no better than the Ramsey plan; thus, $V^r(\gamma,\beta) \geq V(\gamma,\beta)$. Moreover, recalling that μ is the minimum transition probability, note that the difference between this plan and autarky satisfies

(4.21) $\quad V(\gamma,\beta) - V^a(\gamma,\beta) \geq \beta\mu\epsilon + \beta^2\mu\epsilon + \beta^3\mu\epsilon + ...$

To understand (4.21), notice that for every string g^t there is a probability of at least μ of hitting γ_i at date $t + 1$. Using (4.20), it follows that the expected discounted value at time zero of the difference between the above plan and the autarky plan from time t to time $t + 1$ is at least $\beta^t\mu\epsilon$. Adding over all such pairs of dates gives (4.21). Since the right side of (4.21) does not depend on the initial state and $\beta\mu\epsilon/(1-\beta)$ converges to infinity as β converges to one, the proposition follows. $\square$

We can combine Proposition 3 with Lemma 2 to establish the main result of this section: For large discount factors, the Ramsey outcomes are sustainable. The basic logic is as follows. Proposition 3 guarantees that after any string of government consumption, there is some state into which zero debt is sold. Notice that if zero debt is sold into a state, then the piece of the Ramsey problem defined over all strings emanating from that state has the same form as the original problem with the zero-debt state acting as an initial node to a "new" Ramsey problem. Lemma 2 guarantees that the difference in the tail utilities between the Ramsey and autarky plans goes to infinity after any such zero-debt state. To prove the proposition, we show that this difference goes to infinity after any sequence of government consumption. The Markov assumption guarantees that after any such sequence the probability of never hitting a zero-debt state is small enough so that the utility difference between these plans is dominated by the utility difference after zero-debt states. Consequently, this difference goes to infinity as the discount factor approaches one. We have, then,

Proposition 4 (The Sustainability of Ramsey Outcomes). Under Assumptions 1 and 2, there is some $\underline{\beta}$ in $(0,1)$ such that for all β in $(\underline{\beta},1)$ the Ramsey outcome is sustainable.

Proof. Let $U^r(g^t)$ denote the value of utility at g^t under the Ramsey plan. From (4.6), it follows that the Ramsey plan is sustainable if, for each g^s,

$$(4.22) \qquad \sum_{t=s}^{\infty} \sum_{g^t \supseteq g^s} \beta^t \mu(g^t)[U^r(g^t) - U^a(g_t)] \geq 0.$$

We can divide this sum over all strings g^t which include g^s into two sets of strings. The first set consists of all those strings into which zero debt is sold together with strings that follow such a string. The second set consists of all those strings g^t for which positive debt has been sold into all substrings $g^r \in g^t$ for $r \geq s$.

Proposition 3 guarantees that for any string g^t, there is always some state at $t + 1$—say, $\gamma(g^t)$—into which zero debt is sold. Each such string $(g^t, \gamma(g^t))$ is like an initial state to a Ramsey problem starting at $\gamma(g^t)$. Lemma 2 then guarantees that the present value of the utility difference over strings following $(g^t, \gamma(g^t))$ goes to infinity as β approaches one. Thus, we are done if we can show that the utility difference over the second set of states is bounded. Since zero debt is always sold into some state, it follows that this utility difference is at most

$$(4.23) \qquad \underline{u} + (1-\mu)\beta\underline{u} + (1-\mu)^2\beta^2\underline{u} + \ldots$$

where $\underline{u}$ is the maximal difference between any two utility values and μ is the minimum transition probability. Since for all β in $(0,1)$ the sum in (4.23) is bounded by the constant $\underline{u}/\mu$, we are done. $\square$

This proposition is reminiscent of the Folk Theorem in repeated games which says that with sufficiently little discounting all individually rational payoffs can be supported by subgame perfect Nash equilibria (see Fudenberg and Maskin 1986). In our setup we have blended features of competitive analysis with game theory and thus technically our model is not a game. In order to explore the relationship between our results and results like the Folk Theorem it is useful to briefly explain how our environment can be mapped into a game (see the working paper version of this paper and Chari and Kehoe 1990 for details). We capture the competitive behavior of private agents by setting up an anonymous game in which there is a continuum of private agents who observe only policies, their own decisions and the average or per-capita outcomes of private decisions. The government observes policies and the average private decisions. Since a game does not have budget constraints we define payoffs by adding to the utilities in our model a function which is zero if the agent's budget constraint is met and some large negative number of it is violated. This

formulation implies that all agents will meet their budget constraints in equilibrium. In Chari and Kehoe (1990) we showed that the set of symmetric perfect Bayesian equilibrium outcomes of the anonymous game are the same as the set of sustainable outcomes.

Notice that the game set up here is quite different from standard repeated games. For one thing it is dynamic and stochastic. But, more important, competitive behavior on the part of private agents implies that all individually rational payoffs cannot be supported even with sufficiently little discounting. Here the set of individually rational payoffs consists of all payoffs between the discounted value of the payoffs obtained from violating the budget constraint and the efficient allocation in an economy with lump sum taxes (the command optimum). Propositions 2 and 4 show that the set of sustainable equilibrium payoffs is strictly smaller then this set, ranging only from the autarky payoffs to Ramsey payoffs. The reason for this difference is the competitive behavior of private agents, which corresponds to anonymity in the game. If we allow all agents to observe all actions then strategies which specify reversion to some bad equilibrium if any single private agent fails to choose the lump sum tax allocation can support the command optimum. Moreover, without anonymity, any individually rational payoff can be supported by an appropriate trigger strategy.

Given these results one might ask whether with little discounting it is possible to support all outcomes that are attainable under commitment. The answer is no. Consider, for example, balancing the budget in each period and chooses taxes on the wrong side of the Laffer curve. This outcome is attainable under commitment but gives utility below autarky and thus by Proposition 2 is not sustainable. In this sense our results are more closely related to those of Fudenberg, Kreps, and Maskin (1990) than the standard results on the Folk Theorem.

5. Examples

Here we present examples that clarify the scope and limitations of the theory. We begin with a parametric example that illustrates several features of sustainable outcomes and their associated utility levels.

Example 1 (Features of Sustainable Outcomes). $U(c,\ell) = c^\alpha (\bar{\ell} - \ell)^{1-\alpha}$ with $\alpha = 0.33$ and $\bar{\ell} = 10$. Let government consumption be i.i.d. and take on two values γ_1 and γ_2 with probabilities $\mu_1 = \mu_2 = 0.5$. Let γ_2 be the initial state. We focus on stationary outcomes in which $c(g^{t-1}, \gamma_i) = c(\gamma_i)$ and $\ell(g^{t-1}, \gamma_i) = \ell(\gamma_i)$ for

all g^{t-1}. Recall that an arbitrary pair of contingent sequences (π, x) is the outcome of a sustainable equilibrium if and only if it is attainable under commitment and satisfies the inequalities in (4.6). An outcome is attainable under commitment if it satisfies feasibility together with (4.4) and (4.5). For our stationary example, (4.4) and (4.5) reduce to $R(\gamma_2) + \beta/(1-\beta)[\mu_1 R(\gamma_1) + \mu_2 R(\gamma_2)] = 0$ and $R(\gamma_1) + \beta/(1-\beta)[\mu_1 R(\gamma_1) + \mu_2 R(\gamma_2)] \geq 0$.

The inequalities in (4.6) reduce to, for $i = 1, 2$,

$$(5.1) \qquad U(\gamma_i) + \frac{\beta}{1-\beta}[\mu_1 U(\gamma_1) + \mu_2 U(\gamma_2)] \geq U^a(\gamma_i) + \frac{\beta}{1-\beta}[\mu_1 U^a(\gamma_1) + \mu_2 U^a(\gamma_2)]$$

where $U(\gamma_i) = U(c(\gamma_i), \ell(\gamma_i))$. We find it convenient to normalize the utility of an allocation by dividing by the utility in autarky. Thus, the normalized utility of an allocation is defined to be

$$(5.2) \qquad \frac{U(\gamma_1) + \beta[\mu_1 U(\gamma_1) + \mu_2 U(\gamma_2)]/(1-\beta)}{U^a(\gamma_1) + \beta[\mu_1 U^a(\gamma_1) + \mu_2 U^a(\gamma_2)]/(1-\beta)}.$$

In Figure 1, we plot the set of normalized utilities resulting from stationary sustainable allocations against the discount factor for $\gamma_1 = 1$ and $\gamma_2 = 2$. Four features are notable. First, for $\beta \leq 0.76$, autarky is the only stationary sustainable allocation. Second, if an allocation is sustainable for some discount factor, then it is sustainable for a larger discount factor. Third, for $\beta \geq 0.83$, all allocations with utilities between autarky and Ramsey are sustainable. Finally, note that the maximum sustainable utility jumps at $\beta = 0.7$. This discontinuity in the maximal sustainable utility level also appears in the example in Chari-Kehoe (1990).

It is interesting to examine how the dynamic incentive constraints in (5.1) affect the sustainable policies. In terms of the debt in our two state i.i.d. example (4.16) simplifies to

$$(5.3) \qquad b(\gamma_i) = \frac{1}{U_c(\gamma_i)}\left[R(\gamma_i) + \frac{\beta}{1-\beta}(\mu_1 R(\gamma_1) + \mu_2 R(\gamma_2))\right]$$

for $i = 1, 2$. Since the economy starts in a high spending state with $g_0 = \gamma_2$ the date 0 budget constraint simplifies to

$$(5.4) \qquad b(g_0) = \frac{1}{U_c(\gamma_2)}\left[R(\gamma_2) + \frac{\beta}{1-\beta}(\mu_1 R(\gamma_1) + \mu_2 R(\gamma_2))\right].$$

Since we have assumed that the inherited debt at date 0, $b(g_0)$, is zero, (5.3) and (5.4) make it clear that the inherited debt in any high spending state γ_2 ("war") is zero, so $b(\gamma_2) = 0$. The debt inherited in any low spending state γ_1 ("peace") is positive whenever the outcome is not the autarky outcome. In Figure 2 we plot the debt inherited in peacetime under the optimal sustainable equilibrium and under the Ramsey equilibrium. The debt inherited in wartime is zero in both equilibria. In Figure 3 we plot the tax rates under both equilibria. For $\beta < 0.76$ the optimal sustainable debt is zero and the tax rates are the autarky ones. For $\beta > 0.83$ the debt and tax rates of the optimal sustainable equilibrium coincide with those in the Ramsey equilibrium. More interestingly, notice that for intermediate discount factors the debt inherited in peacetime is smaller in the optimal sustainable equilibrium. For such discount factors in wartime the optimal sustainable tax rate is higher than the Ramsey tax rate while in peacetime the reverse is true. Thus, when binding, the dynamic incentive constraints force debt to be smaller and taxes to be more volatile than in the Ramsey equilibrium.

We now consider a sequence of mean-preserving spreads of government consumption from $\gamma_1 = \gamma_2 = 1.5$ up through $\gamma_1 = 1$ and $\gamma_2 = 2$. In Figure 4, we plot the set of stationary sustainable utilities against the variance of government consumption at $\beta = 0.9$. Note that the sustainable utility set is monotonically increasing in the variance. Intuitively, as government consumption becomes more variable, the benefits to tax-smoothing increase, so the punishment of reverting to autarky forever becomes more severe. This example highlights the connection between the variability of government consumption and the sustainability of good outcomes.

It is useful to relate this last result to those in the industrial organization literature. For example, in Rotemberg and Saloner's (1986) model of collusion over the business cycle and Green and Porter's (1984) model of collusion with demand uncertainty higher variability in the economy typically makes it more difficult to support good outcomes. (See pages 249 and 264 of Tirole's 1989 book). One difference between our work and this work is that our model is dynamic and the set of physically possible benefits and losses depends on fundamentally intertemporal considerations while in theirs it depends on atemporal considerations. For example, unlike in these models, in our model if there were no variability the best and the worst equilibria would coincide. More generally, in our model increased variability has two effects which work in opposite

directions. To understand these effects notice that if either of the incentive constraints in (5.1) bind it is the one starting in peace. As the variability increases the debt inherited in peacetime in the Ramsey equilibrium increases and thus the current gains from default increase. This increased variability, however, also increases the value in the future of smoothing tax distortions and hence the future losses from defecting increase. For a large class of parametric examples we have found that the future losses outweighed the current gains. Thus we found that increased variability typically makes it easier to support good outcomes.

Next, in proving the Ramsey outcomes were sustainable, we assumed that the fluctuations in government consumption never damped out. To see the importance of this assumption suppose, for example there is some date T such that government consumption is constant after T. Clearly, at T the best the government can do is to default on the debt at T and to balance its budget in every period thereafter. Anticipating this policy consumers will not purchase any debt in period $T - 1$, regardless of the previous history of government policy. The best policy for the government at $T - 1$ is to default on inherited debt and to balance its budget. By induction it follows that in any sustainable equilibrium the debt is always zero and the equilibrium outcome is autarky forever. Thus the Ramsey outcome is not sustainable for any discount factor less than unity. Next, we present an example in which the best sustainable utility levels lie strictly between the autarky and sustainable levels. Consider

Example 2 (A Markov Chain With Absorbing States). Let government consumption follow a three-state Markov chain with states $\{\gamma_1, \gamma_2, \gamma_3\}$ with $\gamma_1 = 0$ and with transition matrix

$$
\begin{bmatrix}
1 & 0 & 0 \\
\mu_{21} & \mu_{22} & \mu_{23} \\
\mu_{31} & \mu_{32} & \mu_{33}
\end{bmatrix}.
$$

Note that $\gamma_1 = 0$ is an absorbing state. Since government consumption is constant once γ_1 is realized we can use the same reasoning as above to show that the debt sold into state γ_1 is always zero. In terms of characterizing the sustainable allocations note that the allocations in state γ_1 are the autarky allocations. Thus the terms involving state γ_1 are the same on both sides of (4.6) and can be eliminated. With stationary allocations the inequalities in (4.6), for, say γ_2, reduce to

$$
U(\gamma_2) + \beta[\mu_{22}U(\gamma_2) + \mu_{23}U(\gamma_3)] + \dots \geq U^a(\gamma_2) + \beta[\mu_{22}U^a(\gamma_2) + \mu_{23}U^a(\gamma_3)] + \dots.
$$

Since the debt sold into state γ_1 is always zero the terms in (4.4) and (4.5) involving γ_1 also drop out. Thus the problem of characterizing the set of sustainable equilibria is very similar to the problem of characterizing the sustainable equilibria in an economy with only states γ_2 and γ_3 (except that the probabilities of these states do not sum to 1). Using arguments similar to those in Proposition 4 we can show that for a high enough discount factor the best sustainable outcomes maximize (4.2) subject to (4.1) and (4.4) with the additional constraint that the debt sold into state γ_1 is zero. It is easy to show that in the Ramsey problem tax distortions are optimally smoothed by selling positive debt into all zero states and raising some revenues in them. Hence, for any discount factor the best sustainable utility levels are bounded away from the Ramsey utility levels. Thus, in this example, the best sustainable utility levels typically lie strictly between the autarky and the Ramsey utility levels.

6. Conclusion

The point of this paper and its companion (Chari and Kehoe 1989) is to carefully analyze whether reputational forces can solve the time inconsistency problem of default in a standard macromodel with government debt. In this paper we showed that if the government cannot enforce private debt claims and if government consumption fluctuates enough forever, then reputational forces can support equilibria with no default. In the companion paper we showed that with enforceable private debt claims standard trigger strategies cannot support positive debt equilibria.

In this pair of papers we focussed on the interaction between the enforceability of private debt claims and the ability to sustain good outcomes. In doing so we assumed there were no problems in collecting taxes. We think this asymmetric formulation is a useful first step in analyzing the time inconsistency problem of default. A more general analysis would introduce the possibility of tax evasion together with enforceability of private debt. We think such an analysis is best undertaken in a model which makes explicit the asymmetries in information between the government and private agents. Such an analysis would require merging the techniques developed in the mechanism design literature with those in the time consistency literature. Carrying out such an analysis is clearly a formidable but valuable task.

For nearly a decade economists, using differing levels of formality, have used reputational arguments for a variety of macroeconomic issues. Almost all of the formal literature, however, has analyzed

these reputational arguments by using the repeated static game models similar to those used in the industrial organization literature. (See, for example, Barro and Gordon 1983 and Backus and Driffill 1985.) While this literature was a useful first step it is not obvious that its insights carry over to standard macromodels which are inherently dynamic. The classic papers on time consistency (including Kydland and Prescott 1977, Calvo 1978, Lucas and Stokey 1983; and Persson, Persson, and Svensson 1987) use standard macromodels with state variables such as capital, money, or debt. These models give rise to dynamic games and not repeated static games. This paper and its companion make it clear that reputational arguments work in a more subtle way in such dynamic games than they do in repeated games. In particular, we show that the arguments depend on the economic features of the underlying environment such as the nature of enforcement technologies and the nature of the stochastic process for government spending. Whether reputational arguments will work in other standard models with money and capital is an open question. If they do, we suspect that they will depend on economic features of the environment. We hope that some of the techniques developed here will be useful in analyzing these questions.

References

ABREU, D. (1988), "On the Theory of Infinitely Repeated Games with Discounting", *Econometrica*, **56**, 383–96.

ATKESON, A. (1991), "International Lending with Moral Hazard and Risk of Repudiation", *Econometrica*, **59**, 1069–90.

BACKUS, D., and DRIFFILL, J. (1985), "Inflation and Reputation", *American Economic Review*, **75**, 530–38.

BARRO, R., and GORDON, D. (1983), "Rules, Discretion and Reputation in a Model of Monetary Policy", *Journal of Monetary Economics*, **12**, 101–21.

BULOW, J., and ROGOFF, K. (1989), "Sovereign Debt: Is to Forgive to Forget?", *American Economic Review*, **79**, 43–50.

CALVO, G. (1978), "On the Time Consistency of Optimal Policy in a Monetary Economy", *Econometrica*, **46**, 1411–28.

CHARI, V., and KEHOE, P. (1989), "Sustainable Plans and Debt" (Research Department Staff Report 125, Federal Reserve Bank of Minneapolis).

CHARI, V., and KEHOE, P. (1990), "Sustainable Plans", *Journal of Political Economy*, **98**, 783–802.

COLE, H., and ENGLISH, W. (1988), "Two-Sided Sovereign Default and International Equity Contracts" (International Economics Research Center Discussion Paper 12, University of Pennsylvania).

EATON, J., GERSOVITZ, M., and STIGLITZ, J. (1986), "The Pure Theory of Country Risk", *European Economic Review*, **30**, 481–513.

GREEN, E., and PORTER, R. (1984), "Non-Cooperative Collusion under Imperfect Price Information", *Econometrica*, **52**, 87–100.

FUDENBERG, D., KREPS, D., and MASKIN, E. (1990), "Repeated Games with Long-Run and Short-Run Players", *Review of Economic Studies*, **57**, 555–74.

FUDENBERG, D., and MASKIN, E. (1986), "Folk Theorems for Repeated Games with Discounting or with Incomplete Information", *Econometrica*, **54**, 533–54.

GROSSMAN, H., and VAN HUYCK, J. (1986), "Sovereign Debt as a Contingent Claim: Excusable Default, Repudiation, and Reputation" (Working Paper 86-27, Brown University). [Published?]

KYDLAND, F., and PRESCOTT, E. (1977), "Rules Rather than Discretion: The Inconsistency of Optimal Plans", *Journal of Political Economy,* **85,** 473-91.

LUCAS, E., and STOKEY, N. (1983), "Optimal Fiscal and Monetary Policy in an Economy without Capital", *Journal of Monetary Economics,* **12,** 55-93.

MANUELLI, R. (1986), "A General Equilibrium Model of International Credit Markets" (mimeo, Stanford University, Palo Alto).

PERSSON, M., PERSSON, T., and SVENSSON, L. (1987), "Time Consistency of Fiscal and Monetary Policy", *Econometrica,* **55,** 1419-31.

PRESCOTT, E. (1977), "Should Control Theory be Used for Economic Stabilization?", in (eds.) Brunner, K. and Meltzer, A. *Optimal Policies, Control Theory and Technology Exports,* Carnegie-Rochester Conference-Series on Public Policy, **7,** 13-38 (Amsterdam: North-Holland).

ROTEMBERG, J., and SALONER, G. (1986), "A Supergame-Theoretic Model of Business Cycles and Price Wars during Booms", *American Economic Review,* **76,** 360-407.

TIROLE, J. (1989), *The Theory of Industrial Organization,* (Cambridge: MIT Press Massachusetts).

WEITZMAN, M. (1973), "Duality Theory for Infinite Horizon Convex Models", *Management Science,* **19,** 783-89.

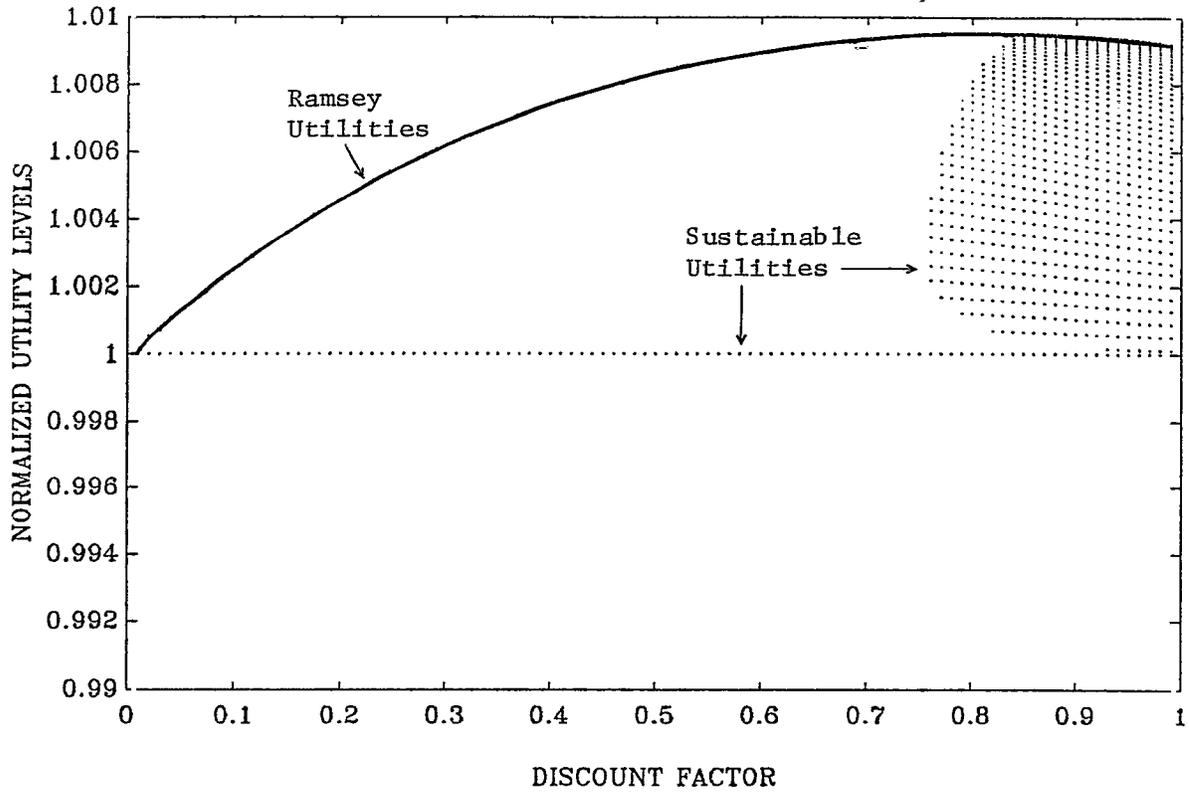Figure 1

Sustainable Utilities vs. The Discount Factor

Figure 2
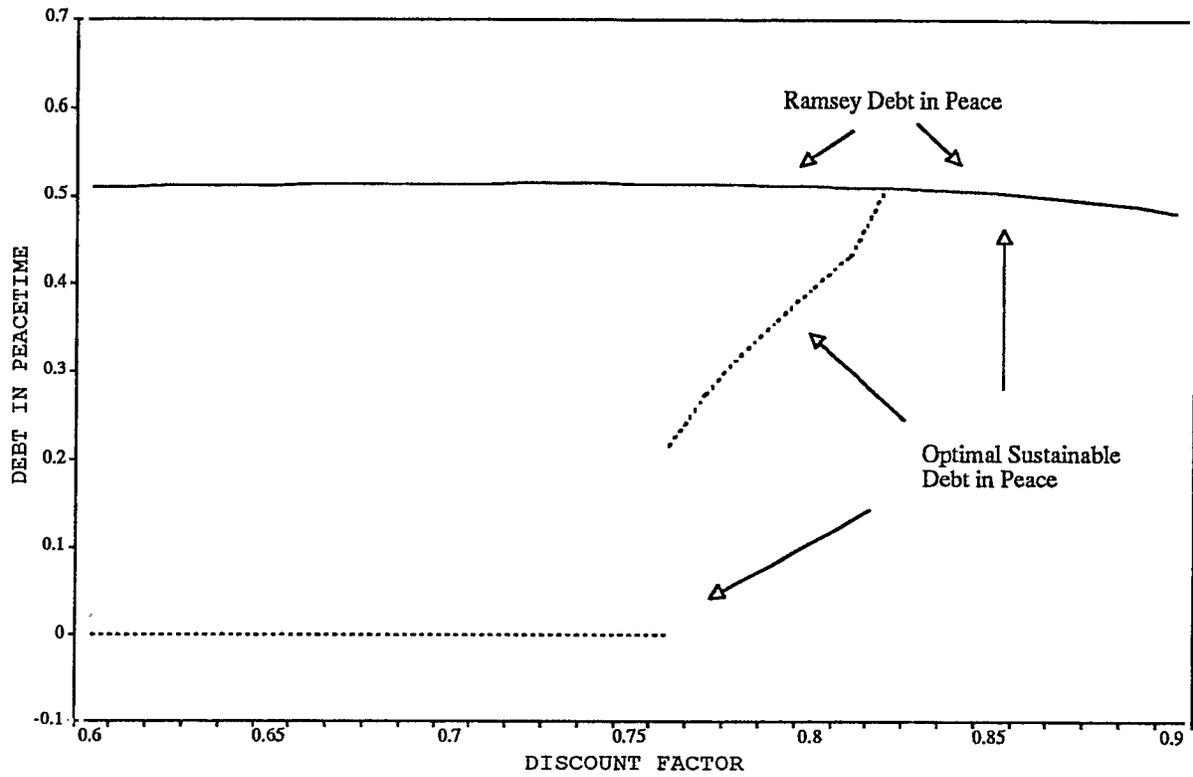
Optimal Sustainable Debt vs. the Discount Factor

Figure 3

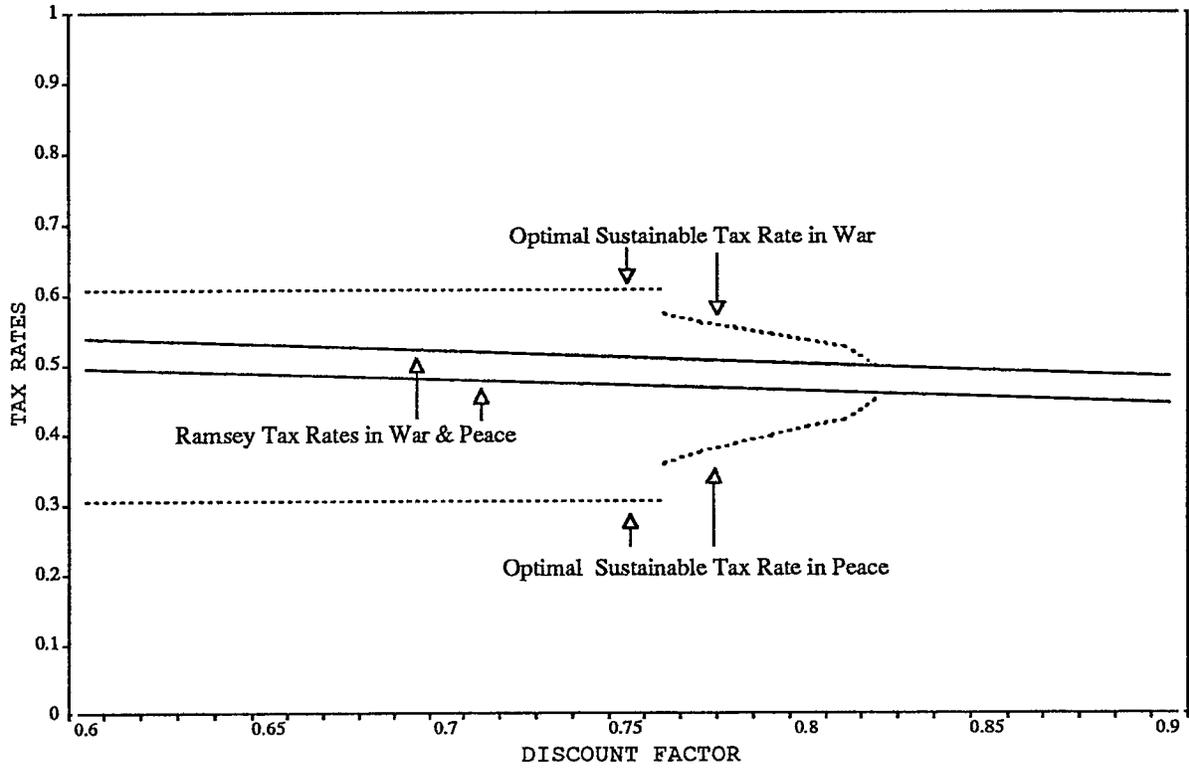### Optimal Sustainable Tax Rates vs. the Discount Factor

Figure 4

Sustainable Utilities vs. The Variance of Government Consumption